SEMI EQUIVELAR MAPS ON THE SURFACE OF EULER CHARACTERISTIC -1

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Abstract

Semi-Equivelar maps are generalizations of Archimedean solids to the surfaces other than 2-sphere. In earlier work a complete classification of semi-equivelar map of type $(3^5, 4)$ on the surface of Euler characteristic -1 was given. In the meantime Karabas an Nedela classified vertex transitive semi-equivelar maps on the double torus. In this article we study the types of semi-equivelar maps on double torus that are also available on the surface of Euler characteristic -1. We classify them and show that none of them are vertex transitive.

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1 Introduction

A triangulation of a surface is called d-covered if each edge of the triangulation is incident with a vertex of degree d. We got interested in studying the content presented in this article while attempting to answer a question of Negami and Nakamoto [17] about existence of d-covered triangulations for closed surfaces. We had answered their question in affirmative [19] for the surfaces of Euler characteristic $-127 \leq \chi \leq -2$ and further became interested in looking at what happens for surfaces with $\chi = -1$. It was here that due to curvature considerations of this surface we had to construct a map on this surface which we named as Semi-equivelar map [23]. Such maps have also been studied in various forms (see [1], [7, 8, 12, 11]). In the meantime we came to learn that Nedela and Karabas [13], [14] have worked along similar lines and classified all the vertex transitive Archimedean maps on orientable surfaces of Euler characteristics -2, -4 and -6 (see also [15]). In particular, they have shown that there are seventeen isomorphism classes of Archimedean maps on orientable surface of Euler characteristic -2, out of which exactly fourteen are semi-equivelar maps with eleven distinct face sequences of types: $(3^5, 4)$, $(3^4, 4^2)$, $(3^4, 8)$, $(3^2, 4, 3, 6)$, $(3, 4^4)$, $(3, 4, 8, 4)$, $(3, 6, 4, 6)$, $(4^3, 6)$, $(4, 6, 16)$, $(4, 8, 12)$, $(6^2, 8)$. An orientable closed surface of Euler characteristic -2 is double cover of non orientable closed surface of Euler characteristic -1. This motivated us to explore the existence of above eleven types of semi-equivelar maps on non orientable surface of Euler characteristic -1. In the article [23] we have classified the semi-equivelar map of the type $(3^5, 4)$ on this surface. Here, we investigate remaining types of semi-equivelar maps on this surface. In next few paragraphs we describe the definitions and terminologies used in this article. These definitions and terminologies are given in [10] and we are giving them here for the sake of ready reference. A standard reference on the subject of polyhedral maps is the article [3] of Brehm and Schulte. For graph theory related terminologies one may also refer to[21] and for topological preliminaries and terminologies one may refer to [20].

Throughout this article the term graph will mean a finite simple graph. A cycle of length m or a *m-Cycle*, usually denoted by C_m, is by definition a connected 2-regular graph with m vertices. A 2-dimensional *Polyhedral Complex* K is a finite collection of m_i-cycles, where $\{m_i\colon 1 \leq i \leq n \text{ and } m_i \geq 3\} \subseteq \mathbb{N}$, together with vertices and edges of the cycles such that the non-empty intersection of any two cycles is either a vertex or is an edge. The cycles are called faces of K. The notations $V(K)$ and $EG(K)$ are used to denote the set of vertices and edges of K respectively. A geometric object, called *geometric career* of K, denoted by $|K|$ can be associated to a polyhedral complex K in the following manner: corresponding to each m-cycle C_m in K, consider a m-gon D_m whose boundary cycle is C_m. Then $|K|$ is union of all such m-gons. The complex K is said to be connected (resp. compact or orientable) if $|K|$ is connected (resp. compact or orientable) topological space. A polyhedral complex K is called a *Polyhedral 2-manifold* if for each vertex v the faces containing v are of the form $C_{m_1}, \ldots, C_{m_p}$ where $C_{m_1} \cap C_{m_2}, \ldots C_{m_{p-1}} \cap C_{m_p}$, and $C_{m_p} \cap C_{m_1}$ are edges for some $p \geq 3$. A connected polyhedral 2-manifold is called a *Polyhedral Map*. We will also use the term *map* for a polyhedral map. Among any two complexes K_1 and K_2 we define an isomorphism to be a bijective map $f\colon V(K_1) \longrightarrow V(K_2)$ for which $f(\sigma)$ is a face in K_2 if and only if σ is a face in K_1. If $K_1 = K_2$ then f is said to be an automorphism of K_1. The set of all automorphisms of a polyhedral complex K form a group under the operation of composition of maps. This group is called the automorphism group of K. If this group acts transitively on the set $V(K)$ then the complex is called a *vertex transitive* complex. Some vertex transitive maps of Euler characteristic 0 have been studied in [4] and many others in [2], [5], [6], [16] and [18].

The *face sequence* (see [23]) of a vertex v in a map is a finite cyclically ordered sequence $(a^p, b^q, ..., m^r)$ of powers of positive integers $a, b, ..., m \geq 3$ and $p, q, ..., r \geq 1$, such that through the vertex v, p numbers of C_a, q numbers of C_b, $\ldots$, r numbers of C_m are incident in the given cyclic order. A map K is said to be *Semi-Equivelar* if face sequence of each vertex of K is same. A SEM with face sequence $(a^p, b^q, ..., m^r)$, is also called SEM of type $(a^p, b^q, ..., m^r)$. In [22], maps with face sequence $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$ have been considered.

Let $EG(K)$ be the edge graph of a map K and $V(K) = \{v_1, v_2, \ldots, v_n\}$. Let $L_K(v_i) = \{v_j \in V(K)\colon v_iv_j \in EG(K)\}$. For $0 \leq t \leq n$ define a graph $G_t(K)$ with $V(G_t(K)) = V(K)$ and $v_iv_j \in EG(G_t(K))$ if $|L_K(v_i) \bigcap L_K(v_j)| = t$. In other words the number of elements in the set $L_K(v_i) \bigcap L_K(v_j)$ is t. This graph was introduced in [6] by B. Datta. Moreover if K and K' are two isomorphic maps then $G_i(K) \cong G_i(K')$ for each i. We have used these graphs in this article to determine whether two SEMs are isomorphic?

In the article [23] it has been shown that:

PROPOSITION **1.1** *There exactly three non isomorphic semi equivelar maps of type* $(3^5, 4)$ *on the surface of Euler characteristic* -1.

$\square$.

In the present article we show:

LEMMA **1.1** *If* K *is a semi-equivelar map of type (3, 4, 8, 4) on the surface of Euler characteristic* -1 *then* K *is isomorphic to* $K_1(3, 4, 8, 4)$ *or* $K_2(3, 4, 8, 4)$ *given in example Section 2.*

LEMMA **1.2** *If* M *is a semi-equivelar map of type* $(4, 6, 16)$ *on the surface of Euler characteristic* -1*, then* M *is isomorphic to* $M_1(4, 6, 16)$ *or* $M_2(4, 6, 16)$ *given in example Section 2.*

LEMMA **1.3** *If* N *is a semi-equivelar map of type* $(6^2, 8)$ *on the surface of Euler characteristic* -1*, then* N *is isomorphic to* $N_1(6^2, 8)$ *or* $N_2(6^2, 8)$ *given in example Section 2.*

Thus from the above Proposition 1.1 and Lemma 1.1, 1.2, 1.3 it follows that:

THEOREM **1.1** *There are at least nine non-isomorphic semi-equivelar maps on the surface of Euler characteristic* -1.

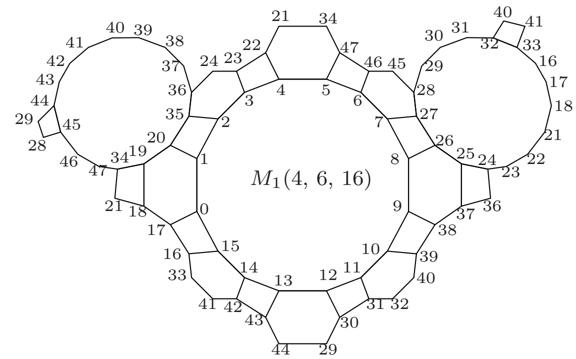
$\square$

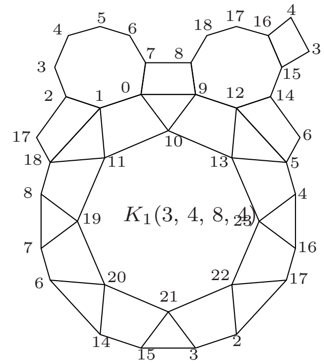
In the article we also show that:

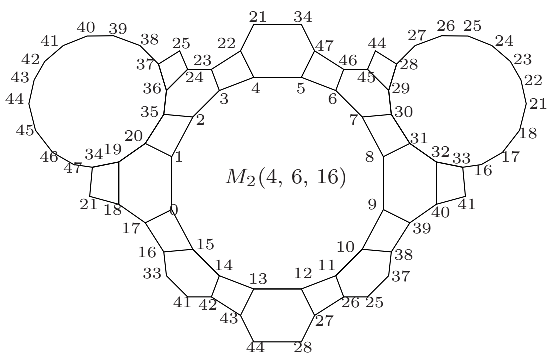
THEOREM **1.2** *There exist no semi-equivelar maps of types* $(3^4, 8)$*,* $(3^2, 4, 3, 6)$*,* $(3, 6, 4, 6)$*,* $(4^3, 6)$ *and* $(4, 8, 12)$ *on the surface of Euler characteristics -1.*

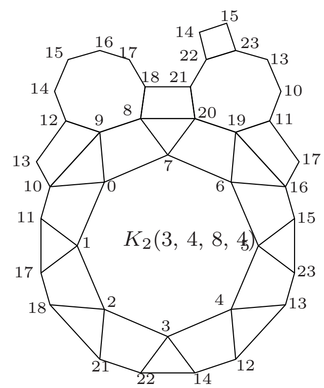
The article is organized in the following manner. In next section, we present examples of semi-equivelar maps on the surface of Euler characteristic -1. In the section, we describe the results and their proofs. We conclude the article by presenting a tabulated list of semi-equivelar maps on the surface of Euler characteristic -1.

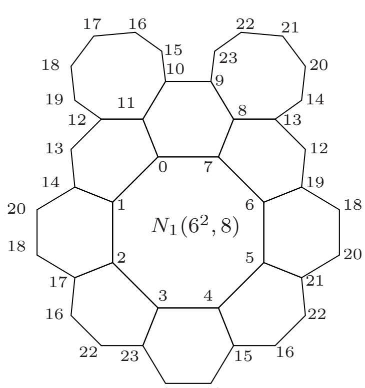
2 Examples: Semi-equivelar maps on the surface of Euler characteristic -1

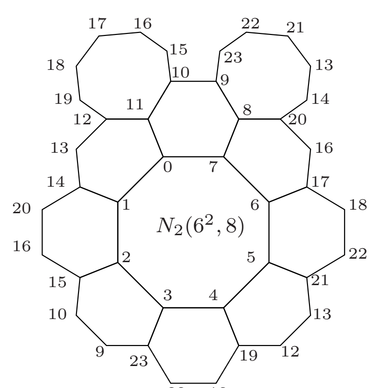












Claim **1** *$N_1(6^2,8) \not\cong N_2(6^2,8)$ and $K_1(3,4,8,4) \not\cong K_2(3,4,8,4)$, also $N_1(6^2,8)$, $N_2(6^2, 8)$, $K_1(3,4,8,4)$ and $K_2(3,4,8,4)$ are not vertex transitive.*

Proof: Consider the graphs $EG(G_{12}(N_1(6^2,8))) = \{[0,7], [3,4], [8,13], [11,12], [15,16], [22,23]\}$, $EG(G_{12}(N_2(6^2,8))) = \{[4,5], [18,19], [21,22]\}$, $EG(G_2(K_1(3,4,8,4))) = C_{12}(1, 10, 12, 6, 19, 18, 2, 21, 14, 5, 23, 17) \cup C_6(4, 13, 9, 7, 20, 15)$ and $EG(G_2(K_2(3,4,8,4))) = C_{21}(0, 8, 21, 3, 12, 10, 1, 18, 20, 6, 15, 22, 2, 17, 19, 7, 9, 13, 5, 16, 11)$. From these graphs and discussions in Chapter 1 (page 12) it is evident that $N_1(6^2,8) \not\cong N_2(6^2,8)$ and $K_1(3,4,8,4) \not\cong K_2(3,4,8,4)$. Also from these graphs one can deduce that above four maps are not vertex transitive. This proves the claim. □

Claim **2** *$M_1(4,6,16) \not\cong M_2(4,6,16)$ and $M_1(4,6,16)$, $M_2(4,6,16)$ are not vertex transitive.*

Proof: Let $A(EG(M_1))$ and $A(EG(M_2))$ denote the adjacency matrices associated to edge graphs of $M_1(4,6,16)$ and $M_2(4,6,16)$, respectively. Let $P_1(x)$ and $P_2(x)$ denote the characteristic polynomials of $A(EG(M_1))$ and $A(EG(M_2))$ respectively. If the map $M_1(4,6,16)$ and $M_2(4,6,16)$ are isomorphic then $P_1(x) = P_2(x)$, (see [16]). We have (using Maple):

$P_1(x) = x^{48} - 73x^{46} + 2454x^{44} - 50419x^{42} + 708648x^{40} - 63x^{39} + 3326x^{37} + 55370675x^{36} - 78998x^{35} - 325536254x^{34} + 1117272x^{33} + 1488079446x^{32} - 10498532x^{31} - 5328759647x^{30} + 69274014x^{29} + 15001009001\, x^{28} - 330979906x^{27} - 33214008513x^{26} + 1164748518x^{25} + 57733175145x^{24} - 3045404365x^{23} - 78484320585x^{22} + 5935770108x^{21} + 82965261974x^{20} - 8621690840x^{19} - 67636071362x^{18} + 9302657658x^{17} + 42014823892x^{16} - 7407374240x^{15} - 19530592234x^{14} + 4302417304x^{13} + 6604154516x^{12} - 1787400560x^{11} - 1549106652x^{10} + 513857976x^9 + 230136488x^8 - 96466160x^7 - 17066976x^6 + 10545344x^5 - 49440x^4 - 495936x^3 + 67264x^2 - 1920x$;

$P_2(x) = x^{48} - 72x^{46} + 2388x^{44} - 48424x^{42} + 672018x^{40} - 28x^{39} - 6770448x^{38} + 1464x^{37} + 51267848x^{36} - 34548x^{35} - 298108536x^{34} + 486936x^{33} + 1348802145x^{32} - 4573164x^{31} - 4785171566x^{30} + 30247956x^{29} + 13360329054x^{28} - 145305100x^{27} - 29376425928x^{26} + 515828328x^{25} + 50783351168x^{24} - 1365657624x^{23} - 68773076142x^{22} + 2706801464x^{21} + 72559583454x^{20} - 4017232620x^{19} - 59173427088x^{18} + 4451481228x^{17} + 36880710516x^{16} - 3658879076x^{15} - 17277557628x^{14} + 2204369472x^{13} + 5931587385x^{12} - 953952300x^{11} - 1432856946x^{10} + 286671228x^9 + 226687857x^8 - 56423208x^7 - 20151768x^6 + 6499968x^5 + 573840x^4 - 330368x^3 + 26880x^2$.

Therefore $M_1(4,6,16) \not\cong M_2(4,6,16)$. Also, we have $EG(G_{15}(M_1(4,6,16))) = EG(G_{15}(M_2(4,6,16))) = C_8(0, 2, 4, 6, 8, 10, 12, 14) \cup C_8(1, 3, 5, 7, 9, 11, 13, 15) \cup C_8(16, 18, 22, 24, 26, 28, 30, 32) \cup C_8(17, 21, 23, 25, 27, 29, 31, 33) \cup C_8(19, 35, 37, 39, 41, 43, 45, 47) \cup C_8(20, 36, 38, 40, 42, 44, 46, 34)$. Let $\alpha \in Aut(M_1(4,6,16))$ such that $\alpha(1) = 2$ then α induces an automorphism on $EG(G_{15}(M_1(4,6,16)))$. So $\alpha\{3,15\} = \{0,4\}$. This implies $\alpha(3) = 0$ or 4. But from the links of 1 and 2 it is easy to see that $\alpha(3) \neq 0$. So we have

$\alpha(3) = 4$, this implies $\alpha(13) = 6$ and $\alpha(35) = 20$. From $\alpha(35) \mapsto 20$ we get $\alpha(42) = 43$. Now considering lk(42) and lk(43) and the map $\alpha(42) \mapsto 43$, we see that $\alpha(13) = 14$, a contradiction. Thus there is no automorphism which maps 1 to 2. Hence $M_1(4, 6, 16)$ is not vertex transitive. Similarly for $M_2(4, 6, 16)$ we get no automorphism such that $\alpha(1) = 2$. This proves the Claim 2. □

3 Enumeration of SEMs on the surface of Euler characteristic -1

Considering Euler characteristic equation, it is easy to see that semi-equivelar maps of types $(3^4, 4^2)$ and $(3, 4^4)$ do not exist on the surface of Euler characteristic -1. As, in these cases number of vertices required to complete a link of a vertex are more than the number of vertices of the SEMs. From the study of remaining eight types: $(3^4, 8)$, $(3^2, 4, 3, 6)$, $(3, 4, 8, 4)$, $(3, 6, 4, 6)$, $(4^3, 6)$, $(4, 6, 16)$, $(4, 8, 12)$ and $(6^2, 8)$, we show the following :

Lemma **3.1** *There exists no SEM of type* $(3^4, 8)$ *on the surface of Euler characteristic* -1.

Proof: Let M be a SEM of type $(3^4, 8)$ on the surface of Euler characteristic -1. The notation $\mathrm{lk}(i) = C_{10}([i_1, i_2, i_3, i_4, i_5, i_6, i_7], i_8, i_9, i_{10})$ for the link of a vertex i will mean that $[i, i_1, i_{10}]$, $[i, i_9, i_{10}]$, $[i, i_8, i_9]$, $[i, i_7, i_8]$ form triangular faces and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7]$ forms an octagonal face. If $|V|$ denotes the number of vertices in $V(M)$, $E(M)$ denotes the number of edges, $T(M)$ denotes the number of triangular faces and $O(M)$ denotes the number of octagonal faces in map M, respectively, then it is easy to see that $E(M) = \frac{5|V|}{2}$, $T(M) = \frac{4|V|}{3}$ and $O(M) = \frac{|V|}{8}$. By Euler's equation we get, $-1 = |V| - \frac{5|V|}{2} + (\frac{4|V|}{3} + \frac{|V|}{8})$, *i.e.* $-1 = |V|(\frac{-1}{24})$. From the equation we see, if the the map exists then $|V| = 24$. Let $V = V(M) = \{0, 1, \ldots, 23\}$. Now, we prove the lemma by exhaustive search for all M.

Assume without loss of generality that $\mathrm{lk}(0) = C_{10}([1, 2, 3, 4, 5, 6, 7], 8, 9, 10)$. This implies $\mathrm{lk}(7) = C_{10}([6, 5, 4, 3, 2, 1, 0], 8, a, b)$ for some $a, b \in V$. One can see that $(a, b) \in \{(10, 9), (11, 12)\}$. In the first case when $(a, b) = (10, 9)$ then considering lk(10) we see that 1 lies in two octagonal faces, which is not allowed. In second case when $(a, b) = (11, 12)$ then we get $\mathrm{lk}(7) = C_{10}([0, 1, 2, 3, 4, 5, 6], 12, 11, 8)$, $\mathrm{lk}(6) = C_{10}([7, 0, 1, 2, 3, 4, 5], 14, 13, 12)$, $\mathrm{lk}(5) = C_{10}([6, 7, 0, 1, 2, 3, 4], 16, 15, 14)$, $\mathrm{lk}(4) = C_{10}([5, 6, 7, 0, 1, 2, 3], 18, 17, 16)$, $\mathrm{lk}(3) = C_{10}([4, 5, 6, 7, 0, 1, 2], 20, 19, 18)$, $\mathrm{lk}(2) = C_{10}([3, 4, 5, 6, 7, 0, 1], 22, 21, 20)$ and $\mathrm{lk}(1) = C_{10}([2, 3, 4, 5, 6, 7, 0], 10, 23, 22)$. This implies $\mathrm{lk}(8) = C_{10}([9, c, d, e, f, g, h], 11, 7, 0)$ or $\mathrm{lk}(8) = C_{10}([c, d, e, f, g, h, 11], 7, 0, 9)$ for some $c, d, e, f, g, h \in V$. But these two are isomorphic by the map $(0, 7)(1, 6)(2, 5)(3, 4)(9, 11)(10, 12)(13, 23)(14, 22)(15, 21)(16, 20)(17, 19)$. Therefore, it is enough to consider $\mathrm{lk}(8) = C_{10}([9, c, d, e, f, g, h], 11, 7, 0)$. Then we obtain the partial picture of the map M as shown in Figure I. Let $V(O_i)$, for $i = 1, 2, 3$, denote the vertex set of an octagonal face O_i then it is easy to see that $V(O_1) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $V(O_2) = \{8, 9, 13, 14, 17, 18, 21, 22\}$ and $V(O_3) = \{10, 11, 12, 15, 16, 19, 20, 23\}$. In this case we see that $(h, g) \in \{(17, 18), (21, 22)\}$.

If $(h, g) = (17, 18)$ then $\mathrm{lk}(8) = C_{10}([9, c, d, e, f, 18, 17], 11, 7, 0)$, $\mathrm{lk}(17) = C_{10}([8, 9, c, d, e, f, 18], 4, 16, 11)$ and $\mathrm{lk}(18) = C_{10}([17, 8, 9, c, d, e, f], 19, 3, 4)$, where $f \in \{13,\ 21\}$. If $f = 13$ then $e = 14$ and $(c, d) \in \{(21,\ 22),\ (22,\ 21)\}$. In case $(c, d) = (21, 22)$, successively considering lk(18), lk(13) and lk(14) we get $\deg(22) > 5$. A contradiction. On the other hand when $(c, d) = (22, 21)$ then successively considering lk(18), lk(13), lk(14), lk(21), lk(22), lk(9), lk(8) and lk(17), we get $C_4(0, 1, 23, 9) \subseteq \mathrm{lk}(10)$. Again, a contradiction. Also for $f = 21$, considering lk(18) we see lk(21) can not be completed. So $(h, g) \neq (17, 18)$.

If $(h, g) = (21, 22)$ then $f \in \{13, 17\}$. In the first case when $f = 13$ then we have $e = 14$ and $(c, d) = (18, 17)$, now considering lk(14) and lk(17) successively we get a triangular face [15, 16, 17] in M. This is not possible. So $f \neq 13$. On the other hand when $f = 17$ then $e = 18$ and $(c, d) = (14, 13)$, now successively considering lk(18), lk(13), lk(14), lk(9), lk(8), lk(21), lk(22) and lk(17), one can see that lk(10) can not be completed. So $(h, g) \neq (21, 22)$ and thus the lemma is proved. □

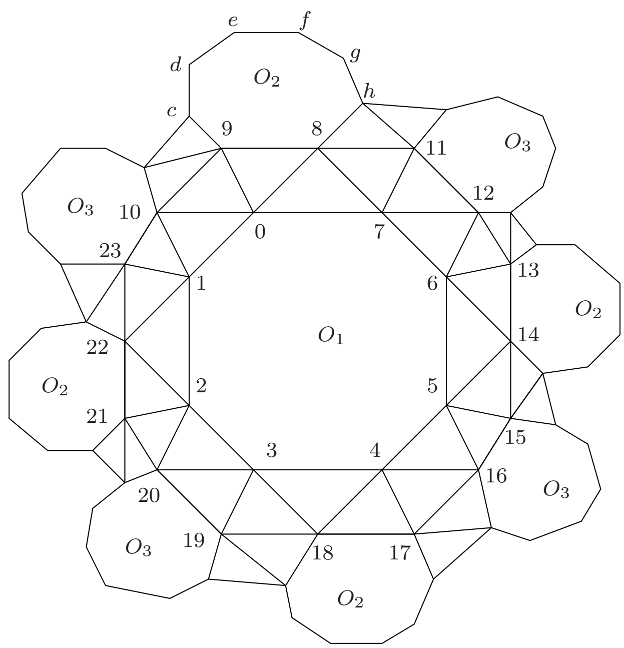


Figure I: Semi-equivelar map M of type $(3^4, 8)$

Lemma **3.2** *There exists no SEM of type* $(3^2, 4, 3, 6)$ *on the surface of Euler characteristic* -1.

Proof : Let G be a SEM of type $(3^2, 4, 3, 6)$ on the surface of Euler characteristic -1. The notation $\mathrm{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], i_6, i_7, i_8, i_9)$ for the link of i will mean that $[i, i_1, i_9]$, $[i, i_5, i_6]$, $[i, i_6, i_7]$ form triangular faces, $[i, i_7, i_8, i_9]$ forms quadrangular face and $[i, i_1, i_2, i_3, i_4\ , i_5]$ forms hexagonal face. Let $|V|$ denote the number of vertices in

$V(G)$. If $E(G)$, $T(G)$, $Q(G)$ and $H(G)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of hexagonal faces in the map G, respectively, then it is easy to see that $E(G) = \frac{5|V|}{2}$, $T(G) = \frac{3|V|}{3}$, $Q(G) = \frac{|V|}{4}$ and $H(G) = \frac{|V|}{6}$. By Euler's equation we see, if the map exists then $|V| = 12$. Let $V = V(G) = \{0, 1, \ldots, 11\}$. Now, we prove the lemma by exhaustive search for all G. Assume that $\mathrm{lk}(0) = C_{11}([1, 2, 3, 4, 5], 6, 7, 8, 9)$ then $\mathrm{lk}(7) = C_{11}([a, b, c, d, e], 6, 0, 9, 8)$ or $\mathrm{lk}(7) = C_{11}([6, a, b, c, d], e, 8, 9, 0)$ for some $a, b, c, d, e \in V$. But, for both the cases we need more than twelve vertices to complete $\mathrm{lk}(7)$. This is not allowed. So the map does not exist. $\square$

Lemma **3.3** *There exists no SEM of type* $(3, 6, 4, 6)$ *on the surface of Euler characteristic* -1.

Proof: Let E be a SEM of type $(3, 6, 4, 6)$ on the surface of Euler characteristic -1. The notation $\mathrm{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], [i_6, i_7, i_8, i_9, i_{10}], i_{11})$ for the link of i will mean that $[i, i_5, i_6]$ forms triangular face, $[i, i_1, i_{11}, i_{10}]$ forms quadrangular face and $[i, i_1, i_2, i_3, i_4, i_5]$, $[i, i_6, i_7, i_8, i_9, i_{10}]$ form hexagonal faces. Let $|V|$ denote the number of vertices in $V(E)$. If $E(E)$, $T(E)$, $Q(E)$ and $H(E)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of hexagonal faces, respectively, then we see that $E(E) = \frac{4|V|}{2}$, $T(E) = \frac{|V|}{3}$, $Q(E) = \frac{|V|}{4}$ and $H(E) = \frac{2|V|}{6}$. By Euler's equation we see, if the map exists then $|V| = 12$. For this, let $V = V(E) = \{0, 1, \ldots, 11\}$. Now, we prove the lemma by exhaustive search for all E. For this assume that $\mathrm{lk}(0) = C_{11}([1, 2, 3, 4, 5], [6, 7, 8, 9, 10], 11)$, then $\mathrm{lk}(1) = C_{11}([0, 5, 4, 3, 2], [a,\ b, c, d, 11], 10)$ for some $a, b, c, d \in V$. Now, it is easy to see that $\mathrm{lk}(1)$ can not be completed, as a, b, c, d have no suitable values in $V(E)$. Therefore the required map does not exist. Thus the lemma is proved. $\square$

Lemma **3.4** *There exists no SEM of type* $(4^3, 6)$ *on the surface of Euler characteristic* -1.

Proof: Let F be a SEM of type $(4^3, 6)$ on the surface of Euler characteristic -1. The notation $\mathrm{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], i_6, i_7, i_8, i_9, i_{10})$ for the link of i will mean that $[i, i_1, i_{10}, i_9]$, $[i, i_5, i_6, i_7]$, $[i, i_7, i_8, i_9]$ form quadrangular faces and $[i, i_1, i_2, i_3, i_4, i_5]$ forms hexagonal face. Let $|V|$ denote the number of vertices in $V(F)$. If $E(F)$, $Q(F)$ and $H(F)$ denote the number of edges, number of quadrangular faces and number of hexagonal faces, respectively, then $E(F) = \frac{4|V|}{2}$, $Q(F) = \frac{3|V|}{4}$ and $H(F) = \frac{|V|}{6}$. By Euler's equation we see if the map exists then $|V| = 12$. For this, let $V = V(F) = \{0, 1, \ldots, 11\}$. Now we prove the lemma by exhaustive search for all F. Assume that $\mathrm{lk}(0) = C_{11}([1, 2, 3, 4, 5], 6, 7, 8, 9, 10)$. This implies, $\mathrm{lk}(7) = C_{11}([b, c, d, e, 6], 5, 0, 9, 8, a)$ or $\mathrm{lk}(7) = C_{11}([b, c, d, e, 8], 9, 0, 5, 6, a)$ for some $a, b, c, d, e \in V$. Then for both the cases of $\mathrm{lk}(7)$ we need more than twelve vertices to complete. But this is not allowed. So we do not get the required map. Thus the lemma is proved. $\square$

LEMMA **3.5** *There exists no SEM of type* $(4, 8, 12)$ *on the surface of Euler characteristic* -1.

Proof: Let M be a SEM of type $(4, 8, 12)$ on the surface of Euler characteristic -1. The notation $\text{lk}(i) = C_{18}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, \boldsymbol{i_{11}}], i_{12}, [\boldsymbol{i_{13}}, i_{14}, i_{15}, i_{16}, i_{17}, i_{18}])$ for the link of i will mean that $[i, i_{11}, i_{12}, i_{13}]$, $[i, i_1, i_{18}, i_{17}, i_{16}, i_{15}, i_{14}, i_{13}]$ and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, i_{11}]$ form 4-gonal, 8-gonal and 12-gonal faces. If $|V|$, $E(M)$, $Q(M)$, $O(M)$ and $R(M)$ denote the number of vertices, number of edges, number of 4-gonal faces, number of 8-gonal faces and number of 12-gonal faces in M, respectively, then we see that $E(M) = \frac{3|V|}{2}$, $Q(M) = \frac{|V|}{4}$, $O(M) = \frac{|V|}{8}$ and $R(M) = \frac{|V|}{12}$. By Euler's equation we see, if the map exists then $|V| = 24$. For this, let $V = V(M) = \{0, 1, \ldots, 23\}$. Now we proceed as follows. Assume that $\text{lk}(0) = C_{18}([\mathbf{1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathbf{11}], 12, [\mathbf{13}, 14, 15, 16, 17, 18])$. This implies $\text{lk}(1) = C_{18}([\mathbf{0}, 11, 10, 9, 8, 7, 6, 5, 4, 3, \mathbf{2}], 19, [\mathbf{18}, 17, 16, 15, 14, 13])$ and $\text{lk}(2) = C_{18}([\mathbf{3}, 4, 5, 6, 7, 8, 9, 10, 11, 0, \mathbf{1}], 18, [\mathbf{19}, a, b, c, d, e])$ for some $a, b, c, d, e \in V$. Observe that $a \in \{12, 20\}$. If $a = 12$ then $b = 13$, for otherwise $\deg(12) > 3$. But, then 13 appears in two octagonal faces, which is not allowed. So we have $a = 20$, this implies $b \in \{12, 21\}$. If $b = 12$ then $c = 13$ and we get 13 in two octagonal faces. So $b = 21$, this implies $c \in \{12, 22\}$. In case $c = 12$, $d = 13$. This implies 13 appears in two octagonal faces. If $c = 22$ then $d = 23$, now we see that e has no suitable value in V so that $\text{lk}(2)$ can be completed. So, the required map does not exist.

Proof of Theorem 1.2: The proof of Theorem 1.2 follows from Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5. □

Proof of Lemma 1.1: Let K be a SEM of type $(3, 4, 8, 4)$ on the surface of Euler characteristic -1. The notation $\text{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5, i_6, i_7], i_8, [i_9, i_{10}], i_{11})$ for the link of i will mean that $[i, i_9, i_{10}]$ forms triangular face, $[i, i_7, i_8, i_9]$, $[i, i_1, i_{11}, i_{10}]$ form quadrangular faces and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7]$ forms octagonal face. Let $|V|$ denote the number of vertices in $V(K)$. If $E(K)$, $T(K)$, $Q(K)$ and $O(K)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of octagonal faces in the map K, respectively, then we see that $E(K) = \frac{4|V|}{2}$, $T(K) = \frac{|V|}{3}$, $Q(K) = \frac{2|V|}{4}$ and $H(K) = \frac{|V|}{8}$. By Euler's equation we see, if the map exists then $|V| = 24$. Let $V = V(K) = \{0, 1, \ldots, 23\}$. Now, we prove the result by exhaustive search for all K.

Let $\text{lk}(0) = C_{11}([1, 2, 3, 4, 5, 6, 7], 8, [9, 10], 11)$, this implies $\text{lk}(9) = C_{11}([b, c, d, e, f, g, 8], 7, [0, 10], a)$ and $\text{lk}(10) = C_{11}([11, l, k, j, i, h, a], 12, [9, 0], 1)$ for some $a, b, c, d, e, f, g, h, i, j, k, l \in V$. Observe that $b = 12$ and $a = 13$, then octagonal faces of the map K are, $O_1 = [0, 1, 2, 3, 4, 5, 6, 7]$, $O_2 = [8, 9, 12, c, d, e, f, g]$ and $O_3 = [13, 10, 11, l, k, j, i, h]$. As, these faces share no vertex with each other, successively we get $c = 14$, $d = 15$, $e = 16$, $f = 17$, $g = 18$, $l = 19$, $k = 20$, $j = 21$, $i = 22$ and $h = 23$. This implies $\text{lk}(9) = C_{11}([12, 14, 15, 16, 17, 18, 8], 7, [0, 10], 13)$, $\text{lk}(10) = C_{11}([11, 19, 20, 21, 22, 23, 13], 12, [9, 0], 1)$ and $\text{lk}(8) = C_{11}([18, 17, 16, 15, 14, 12, 9], 0, [7, x], y)$ for some $x, y \in V$. In this case, $(x, y) \in \{(19, 11), (19, 20), (20, 19), (20, 21), (21, 20), (21, 22), (22, 21), (22, 23), (23, 13), (23, 22)\}$. If $(x, y) = (23, 13)$ then considering $\text{lk}(8)$ and $\text{lk}(13)$ successively we see 12 18 as an edge and a non-edge both. Also, $(19,20) \cong (23,13)$; $(20,19) \cong (22,21)$ and $(20,21) \cong (22,23)$ by the map (0, 9)(1, 12)(2, 14)(3, 15)(4, 16)(5, 17) (6, 18)(7, 8)(11, 13)(19, 23)(20, 22); $(20,19) \cong (21,22)$ by the map (0, 8)(1, 18)(2, 17)(3, 16)(4, 15)(5, 14)(6, 12)(7, 9)(10, 21) (11, 22)(13, 20)(19, 23); $(19,11) \cong (21,20)$ by the map (0, 8)(1, 18)(2, 17)(3, 16)(4, 15)(5, 14)(6, 12)(7, 9)(10, 21)(11, 20)(13, 22). So, it is enough to search the map for $(x,y) \in \{(19, 11), (20, 19), (20, 21), (23, 22)\}$.

When $(x,y) = (20,19)$ then $\mathrm{lk}(8) = C_{11}([9,12,14,15,16,17,18],19,[20,7],0)$, $\mathrm{lk}(7) = C_{11}([0,1,2,3,4,5,6],21,[20,8],9)$, $\mathrm{lk}(20) = C_{11}([21,22,23,13,10,11,19],18, [8,7],6)$ and $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],7,[6,m],n)$ for some $m,n \in V$. In this case $(m,n) \in \{(14, 15), (15, 14), (15, 16), (16, 15), (16, 17), (17, 16)\}$. But $(14,15) \cong (16,15)$ by the map (0, 6)(1, 5)(2, 4)(8, 20)(9, 21)(10, 16)(11, 17)(12, 22)(13, 15)(14, 23)(18, 19), so we consider the following subcases.

When $(m,n) = (15,14)$ then successively considering lk(21), lk(15), lk(6) and lk(16) one can see that lk(17) can not be completed. When $(m,n) = (15,16)$ then completing lk(21), lk(15) and lk(6) we get $\mathrm{lk}(5) = C_{11}([4,3,2,1,0,7,6],15,[14,23],p)$ for some $p \in V$. Observe that, $p \in \{13, 22\}$. But, for both values of p considering lk(14) and lk(23) successively we see that lk(12) can not be completed. So $(m,n) \neq (15,16)$. When $(m,n) = (16,15)$ then completing lk(21), lk(16) and lk(6), we get $\mathrm{lk}(17) = C_{11}([18,8,9,12,14,15,16],6,[5,23],p)$ for some $p \in V$. Now, proceeding as in previous case, we see that the map does not exist. When $(m,n) = (16,17)$ then $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],7,[6,16], 17)$. Now completing lk(16) and lk(6), we get $\mathrm{lk}(5) = C_{11}([4,3,2,1,0,7,6],16,[15,23],p)$ for some $p \in \{13, 22\}$. In the first case when $p = 13$ then considering lk(5) and lk(13) successively we see that lk(15) can not be completed while for $p = 22$, considering lk(22), lk(15) and lk(13) successively we get $C_9(8,9,10,13,14,15,16,17,18) \subseteq \mathrm{lk}(12)$. A contradiction. So, $(m,n) \neq (16,17)$. When $(m,n) = (17,16)$ then $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],7,[6,17], 16)$. Now successively considering lk(6), lk(17), lk(18), lk(5) and lk(11) we see 14 as an edge and a non-edge both. So $(m,n) \neq (17,16)$. Thus for $(x,y) = (20,19)$ the required map does not exist.

Case 1 : If $(x,y) = (19,11)$ then successively we get $\mathrm{lk}(8) = C_{11}([18,17,16,15,14,12,9],0,[7,19],11)$, $\mathrm{lk}(7) = C_{11}([0,1,2,3,4,5,6],20,[19,8],9)$, $\mathrm{lk}(19) = C_{11}([20,21,22,23,13,10,11],18,[8,7],6)$, $\mathrm{lk}(11) = C_{11}([19,20,21,22,23,13,10],0,[1,18],8)$, $\mathrm{lk}(1) = C_{11}([0,7,6,5,4,3,2],17,[18,11],10)$, $\mathrm{lk}(18) = C_{11}([17,16,15,14,12,9,8],19,[11,1],2)$ and $\mathrm{lk}(6) = C_{11}([5,4,3,2,1,0,7],19,[20,m],n)$ for some $m,n \in V$. In this case, $(m,n) \in \{(14, 12), (14, 15), (15, 14), (15, 16), (16, 15), (16, 17)\}$. If $(m,n) = (16,17)$ then considering lk(17) we see 25 as an edge and a non-edge both and, if $(m,n) = (14,15)$ then considering lk(6) and lk(20) successively we see that lk(21) can not be completed. For the remaining values of (m,n), we have following subcases.

Subcase 1.1 : When $(m,n) = (15,14)$ then $\mathrm{lk}(6) = C_{11}([5,4,3,2,1,0,7],19,[20,15],14)$, $\mathrm{lk}(15) = C_{11}([16,17,18,8,9,12,14],5,[6,20],21)$ and $\mathrm{lk}(20) = C_{11}([21,22,23,13,10,11,19],7,[6,15],16)$. This implies $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],15,[16,o],p)$ for some $o,p \in V$. Observe that $(o,p) \in \{(3, 2), (3, 4), (4, 3), (4, 5)\}$. In case $(o,p) \in \{(3, 2),$

$(3,\ 4)\}$ considering lk(21), lk(16) and lk(3) successively we see that lk(4) or lk(17) can not be completed. If $(o,p)=(4,3)$ then considering lk(21), lk(4), lk(16) successively we see that lk(22) can not be completed. If $(o,p)=(4,5)$ then successively considering lk(21), lk(5) and lk(22), we get $C_9(9,10,11,19,20,21,22,23,12)\subseteq$ lk(13). A contradiction. So, $(m,n)\neq(15,14)$

Subcase 1.2 : If $(m,n)=(14,12)$ then successively we get lk(6) $=C_{11}([5,4,3,2,1,0,7],19,[20,14],12)$, lk(14) $=C_{11}([15,16,17,18,8,9,12],5,[6,20],21)$, lk(20) $=C_{11}([21,22,23,13,10,11,19],7,[6,14],15)$, lk(12) $=C_{11}([14,15,16,17,18,9,10],13,[5,6],14)$, lk(5) $=C_{11}([6,7,0,1,2,3,4],23,[13,12],14)$, lk(13) $=C_{11}([23,22,21,20,19,11,10],9,[12,5],4)$ and lk(21) $=C_{11}([22,23,13,10,11,19,20],14,[15,o],p)$ for some $o,p\in V$. It is easy to see that $(o,p)\in\{(3,\ 2),\ (3,\ 4)\}$. In case $(o,p)=(3,4)$, considering lk(21) and lk(4) successively we get $C_9(3,4,23,13,10,11,19,20,21)\subseteq$ lk(22). A contradiction. So $(o,p)=(3,2)$ then completing successively we get lk(16) $=C_{11}([17,18,8,9,12,14,15],3,[4,23],22)$, lk(22) $=C_{11}([23,13,10,11,19,20,21],3,[2,17],16)$, lk(4) $=C_{11}([5,6,7,0,1,2,3],15,[16,23],13)$, lk (23) $=C_{11}([13,10,11,19,20,21,22],17,[16,4],5)$, lk(17) $=C_{11}([18,8,9,12,14,15,16],23,[22,2],1)$, lk(1) $=C_{11}$ $([2,3,4,5,6,7,0],10,[11,18],17)$ and lk(2) $=C_{11}([3,4,5,6,7,0,1],18,[17,22],21)$. This is $K_1(3,4,8,4)$ as given in Section 2.

Subcase 1.3 : When $(m,n)=(15,16)$ then lk(6) $=C_{11}([7,0,1,2,3,4,5],16,[15,20],19)$. This implies lk(15) $=C_{11}([16,17,18,8,9,12,14],21,[20,6],5)$, lk(20) $=C_{11}([21,22,23,13,10,11,19],7,[6,15],14)$ and lk(14) $=C_{11}([12,9,8,18,17,16,15],20,[21,o],p)$ for some $o,p\in V$. Then, $(o,p)\in\{(3,\ 2),\ (3,\ 4),\ (4,\ 3),\ (4,\ 5)\}$. When $(o,p)\in\{(4,\ 3),\ (4,\ 5)\}$ then successively considering lk(14), lk(4) and lk(21), it is easy to see that lk(22) can not be completed. When $(o,p)=(3,2)$ then lk(14) $=C_{11}([15,16,17,18,8,9,12],2,[3,21],20)$, lk(3) $=C_{11}([4,5,6,7,0,1,2],12,[14,21],22)$, lk(21) $=C_{11}([22,23,13,10,11,19,20],15,[14,3],4)$ and lk(22) $=C_{11}([23,13,10,11,19,20,21],3,[4,q],r)$ for some $q,r\in V$. This implies $q=17$ and $r=16$, now considering lk(22), lk(16), lk(5) and lk(23) successively we see that lk(17) can not be completed. So $(o,p)=(3,4)$ then lk(3) $=C_{11}([4,5,6,7,0,1,2],22,[21,14],12)$, completing successively we get lk(21) $=C_{11}([22,23,13,10,11,19,20],15,[14,3],2)$, lk(2) $=C_{11}([1,0,7,6,5,4,3],21,[22,17],18)$, lk(1) $=C_{11}([2,3,4,5,6,7,0],10,[11,18],17)$, lk(22) $=C_{11}([23,13,10,11,19,20,21],3,[2,17],16)$, lk(17) $=C_{11}([18,8,9,12,14,15,16],23,[22,2],1)$, lk(23) $=C_{11}([13,10,11,19,20,21,22],17,[16,5],4)$, lk(5) $=C_{11}([6,7,0,1,2,3,4],13,[23,16],15)$, lk(16) $=C_{11}([17,18,8,9,12,14,15],6,[5,23],22)$, lk(13) $=C_{11}([10,11,19,20,21,22,23],5,[4,12],9)$, lk(4) $=C_{11}([5,6,7,0,1,2,3],14,[12,13],23)$, lk(12) $=C_{11}([14,15,16,17,18,8,9],10,[13,4],3)$. This is isomorphic to $K_2(3,4,8,4)$, as given in Section 2, by the map (0, 23, 7, 22)(1, 13, 6, 21)(2, 10, 5, 20)(3, 11, 4, 19)(8, 14, 17, 9, 15, 18, 12, 16).

Subcase 1.4 : When $(m,n)=(16,15)$ then successively we get lk(6) $=C_{11}([7,0,1,2,3,4,5],15,[16,20],19)$, lk(20) $=C_{11}([21,22,23,13,10,11,19],7,[6,16],17)$, lk(16) $=C_{11}([17,18,8,9,12,14,15],5,[6,20],21)$, lk(2) $=C_{11}([3,4,5,6,7,0,1],18,[17,21],22)$, lk(17) $=C_{11}([18,8,9,12,14,15,16],20,[21,2],1)$, lk(21) $=C_{11}([22,23,13,10,11,19,20],16,[17,2],3)$ and lk (15) $=C_{11}([14,12,9,8,18,17,16],6,[5,o],p)$ for some $o,p\in V$. Observe that, $(o,p)\in\{(23,13),\ (23,\ 22)\}$. But for $(o,p)=(23,13)$, considering lk(15) and lk(13) successively we get $C_9(8,9,10,13,14,15,16,17,18) \subseteq \mathrm{lk}(12)$. This is a contradiction. On the other hand when $(o,p)=(23,22)$ then $\mathrm{lk}(15)=C_{11}([16,17,18,8,9,12,14],22,[23,5],6)$, $\mathrm{lk}(5)=C_{11}([6,7,0,1,2,3,4],13,[23,15],16)$, $\mathrm{lk}(23)=C_{11}([13,10,11,19,20,21,22],14,[15,5],4)$, completing successively we get $\mathrm{lk}(13)=C_{11}([10,11,19,20,21,22,23],5,[4,12],9)$, $\mathrm{lk}(4)=C_{11}([3,2,1,0,7,6,5],23,[13,12],14)$, $\mathrm{lk}(12)=C_{11}([14,15,16,17,18,8,9],10,[13,4],3)$, $\mathrm{lk}(14)=C_{11}([15,16,17,18,8,9,12],4,[3,22],23)$, $\mathrm{lk}(3)=C_{11}([4,5,6,7,0,1,2],21,[22,14],12)$ and $\mathrm{lk}(22)=C_{11}([23,13,10,11,19,20,21],2,[3,14],15)$. This is isomorphic to $K_1(3,4,8,4)$ by the map (0, 11)(1, 10)(2, 13)(3, 23)(4, 22)(5, 21)(6, 20)(7, 19)(9, 18)(12, 17)(14, 16).

Case 2 : When $(x,y)=(20,21)$ then $\mathrm{lk}(8)=C_{11}([9,12,14,15,16,17,18],21,[20,7],0)$, $\mathrm{lk}(7)=C_{11}([0,1,2,3,4,5,6],19,[20,8],9)$, $\mathrm{lk}(20)=C_{11}([21,22,23,13,10,11,19],6,[7,8],18)$ and $\mathrm{lk}(18)=C_{11}([17,16,15,14,12,9,8],20,[21,m],n)$ for some $m,n\in V$. In this case $(m,n)\in\{(2,\ 1),\ (2,\ 3),\ (3,\ 2),\ (3,\ 4),\ (4,\ 3),\ (4,\ 5),\ (5,\ 4),\ (5,\ 6)\}$. When $(m,n)=(2,3)$ then considering lk(18), lk(2), lk(21) and lk(1) successively we see that 11 22 is simultaneously an edge and a non-edge of K. When $(m,n)=(5,4)$ then considering lk(18), lk(21) and lk(6) successively we see that 19 22 is both an edge and a non-edge of K. So, $(m,n)\neq(2,3),(5,4)$. For the remaining values of (m,n) we have following subcases.

When $(m,n)=(4,5)$ then we have $\mathrm{lk}(18)=C_{11}([8,9,12,14,15,16,17],5,[4,21],20)$, $\mathrm{lk}(4)=C_{11}([5,6,7,0,1,2,3],22,[21,18],17)$, $\mathrm{lk}(21)=C_{11}([22,23,13,10,11,19,20],8,[18,4],3)$ and $\mathrm{lk}(22)=C_{11}([23,13,10,11,19,20,21],4,[3,o],p)$ for some $o,p\in V$. Observe that, $(o,p)\in\{(14,\ 15),\ (15,\ 14),\ (15,\ 16),\ (16,\ 15)\}$. If $(o,p)=(14,15)$ then successively considering lk(22), lk(14), lk(3), lk(12), lk(13) and lk(23) we see that $\deg(1)>4$. A contradiction. If $(o,p)\in\{(15,\ 14),\ (15,\ 16)\}$ then considering lk(22), lk(15) and lk(3) successively we see that lk(16) or lk(2) can not be completed. If $(o,p)=(16,15)$ then considering lk(22), lk(16) and lk(3) successively we see that lk(2) can not be completed. So, $(m,n)\neq(4,5)$. When $(m,n)=(3,2)$ then $\mathrm{lk}(18)=C_{11}([8,9,12,14,15,16,17],2,[3,21],20)$. This implies $\mathrm{lk}(3)=C_{11}([4,5,6,7,0,1,2],17,[18,21],22)$, $\mathrm{lk}(21)=C_{11}([22,23,13,10,11,19,20],8,[18,3],4)$ and $\mathrm{lk}(22)=C_{11}([23,13,10,11,19,20,21],3,[4,o],p)$ for some $o,p\in V$. Observe that, $(o,p)\in\{(14,\ 12),\ (14,\ 15),\ (15,\ 14),\ (15,\ 16),\ (16,\ 15)\}$. Now proceeding further as in previous case we get a contradiction for each value of (o,p). So $(m,n)\neq(3,2)$.

Subcase 2.1 : When $(m,n)=(2,1)$ then successively we get $\mathrm{lk}(18)=C_{11}([8,9,12,14,15,16,17],1,[2,21],20)$, $\mathrm{lk}(21)=C_{11}([22,23,13,10,11,19,20],8,[18,2],3)$, $\mathrm{lk}(2)=C_{11}([3,4,5,6,7,0,1],17,[18,21],22)$, $\mathrm{lk}(1)=C_{11}([2,3,4,5,6,7,0],10,[11,17],18)$, $\mathrm{lk}(11)=C_{11}([19,20,21,22,23,13,10],0,[1,17],16)$, $\mathrm{lk}(17)=C_{11}([18,8,9,12,14,15,16],19,[11,1],2)$ and $\mathrm{lk}(3)=C_{11}([4,5,6,7,0,1,2],21,[22,o],p)$ for some $o,p\in V$. In this case we have $(o,p)\in\{(14,\ 12),\ (14,\ 15),\ (15,\ 14)\}$.

If $(o,p)=(14,15)$ then $\mathrm{lk}(3)=C_{11}([4,5,6,7,0,1,2],21,[22,14],15)$,now completing lk(14) and lk(22) we get $\mathrm{lk}(23)=C_{11}([13,10,11,19,20,21,22],14,[12,5],r)$ for some $r\in V$. It is easy to see that $r\in\{4,\ 6\}$. If $r=4$ then successively considering lk(23), lk(4) and lk(15) we get $\deg(13)>4$ and if $r=6$ then considering lk(23) and lk(6) successively we see that 13 19 is both an edge and a non-edge of K. So $(o,p)\neq(14,15)$. When $(o,p)=(15,14)$ then $\mathrm{lk}(3)=C_{11}([4,5,6,7,0,1,2],21,[22,15],14)$, completing lk(15) and lk(22) we get $\mathrm{lk}(23) = C_{11}([13, 10, 11, 19, 20, 21, 22], 15, [16, 5], r)$ for some $r \in V$. Observe that $r \in \{4, 6\}$. Now proceeding as in previous case, we get a contradiction for each value of r. So $(o, p) \neq (15, 14)$.

If $(o, p) = (14, 12)$ then we see that $\mathrm{lk}(3) = C_{11}([4, 5, 6, 7, 0, 1, 2], 21, [22, 14], 12)$, $\mathrm{lk}(14) = C_{11}([15, 16, 17, 18, 8, 9, 12], 4, [3, 22], 23)$, now completing successively we get $\mathrm{lk}(12) = C_{11}([14, 15, 16, 17, 18, 8, 9], 10, [13, 4], 3)$, $\mathrm{lk}(13) = C_{11}([10, 11, 19, 20, 21, 22, 23], 5, [4, 12], 9)$, $\mathrm{lk}(4) = C_{11}([5, 6, 7, 0, 1, 2, 3], 14, [12, 13], 23)$, $\mathrm{lk}(23) = C_{11}([13, 10, 11, 19, 20, 21, 22], 14, [15, 5], 4)$, $\mathrm{lk}(5) = C_{11}([6, 7, 0, 1, 2, 3, 4], 13, [23, 15], 16)$, $\mathrm{lk}(16) = C_{11}([17, 18, 8, 9, 12, 14, 15], 5, [6, 19], 11)$, $\mathrm{lk}(6) = C_{11}([7, 0, 1, 2, 3, 4, 5], 15, [16, 19], 20)$ and $\mathrm{lk}(19) = C_{11}([20, 21, 22, 23, 13, 10, 11], 17, [16, 6], 7)$. This is $K_2(3, 4, 8, 4)$ as given in Section 2.

Subcase 2.2 : When $(m, n) = (3, 4)$ then successively we get $\mathrm{lk}(18) = C_{11}([8, 9, 12, 14, 15, 16, 17], 4, [3, 21], 20)$, $\mathrm{lk}(3) = C_{11}([4, 5, 6, 7, 0, 1, 2], 22, [21, 18], 17)$, $\mathrm{lk}(21) = C_{11}([22, 23, 13, 10, 11, 19, 20], 8, [18, 3], 2)$ and $\mathrm{lk}(17) = C_{11}([16, 15, 14, 12, 9, 8, 18], 3, [4, o], p)$ for some $o, p \in V$. In this case $(o, p) \in \{(13,\ 23),\ (19,\ 11),\ (23,\ 13),\ (23,\ 22)\}$. If $(o, p) = (13, 23)$ then considering lk(17) and lk(13) successively we see that $12\,17$ is both an edge and a non-edge of K. If $(o, p) = (19, 11)$ then successively considering lk(17), lk(11) and lk(1) we see easily that lk(4) can not be completed. If $(o, p) = (23, 13)$ then considering lk(17) and lk(13) we see that $12\,16$ is both an edge and a non-edge of K. If $(o, p) = (23, 22)$ then $\mathrm{lk}(17) = C_{11}([18, 8, 9, 12, 14, 15, 16], 22, [23, 4], 3)$. This implies $\mathrm{lk}(22) = C_{11}([23, 13, 10, 11, 19, 20, 21], 3, [2, 16], 17)$, completing successively we get $\mathrm{lk}(4) = C_{11}([5, 6, 7, 0, 1, 2, 3], 18, [17, 23], 13)$, $\mathrm{lk}(13) = C_{11}([10, 11, 19, 20, 21, 22, 23], 4, [5, 12], 9)$, $\mathrm{lk}(12) = C_{11}([14, 15, 16, 17, 18, 8, 9], 10, [13, 5], 6)$, $\mathrm{lk}(6) = C_{11}([7, 0, 1, 2, 3, 4, 5], 12, [14, 19], 20)$, $\mathrm{lk}(19) = C_{11}([20, 21, 22, 23, 13, 10, 11], 15, [14, 6], 7)$, $\mathrm{lk}(14) = C_{11}([15, 16, 17, 18, 8, 9, 12], 5, [6, 19], 11)$, $\mathrm{lk}(5) = C_{11}([6, 7, 0, 1, 2, 3, 4], 23, [13, 12], 14)$, $\mathrm{lk}(23) = C_{11}([13, 10, 11, 19, 20, 21, 22], 16, [17, 4], 5)$, $\mathrm{lk}(2) = C_{11}([3, 4, 5, 6, 7, 0, 1], 15, [16, 22], 21)$, $\mathrm{lk}(15) = C_{11}([16, 17, 18, 8, 9, 12, 14], 19, [11, 1], 2)$, $\mathrm{lk}(11) = C_{11}([19, 20, 21, 22, 23, 13, 10], 0, [1, 15], 14)$, $\mathrm{lk}(1) = C_{11}([2, 3, 4, 5, 6, 7, 0], 10, [11, 15], 16)$ and $\mathrm{lk}(16) = C_{11}([17, 18, 8, 9, 12, 14, 15], 1, [2, 22], 23)$. This is isomorphic to $K_2(3, 4, 8, 4)$ by the map (0, 20, 16, 3, 23, 12)(1, 21, 15, 2, 22, 14)(4, 13, 9, 7, 19, 17)(5, 10, 8, 6, 11, 18).

Subcase 2.3 : When $(m, n) = (4, 3)$ then successively we get $\mathrm{lk}(18) = C_{11}([8, 9, 12, 14, 15, 16, 17], 3, [4, 21], 20)$, $\mathrm{lk}(4) = C_{11}([5, 6, 7, 0, 1, 2, 3], 17, [18, 21], 22)$, $\mathrm{lk}(21) = C_{11}([22, 23, 13, 10, 11, 19, 20], 8, [18, 4], 5)$ and $\mathrm{lk}(22) = C_{11}([23, 13, 10, 11, 19, 20, 21], 4, [5, o], p)$ for some $o, p \in V$. Then we see that $(o, p) \in \{(14,\ 15),\ (15,\ 14),\ (15,\ 16),\ (16,\ 15),\ (16,\ 17)\}$. If $(o, p) = (14, 15)$ then successively considering lk(22), lk(14), lk(6) we see that lk(12) can not be completed. If $(o, p) = (15, 14)$ then successively considering lk(22), lk(15), lk(6), lk(19) and lk(17) we see that lk(11) can not be completed. If $(o, p) = (15, 16)$ then successively considering lk(15), lk(6), lk(19) and lk(12) we see that $11\,13$ is both an edge and a non-edge of K. If $(o, p) = (16, 15)$ then considering lk(22), lk(16) and lk(6) successively we get $\deg(17) > 4$. If $(o, p) = (16, 17)$ then $\mathrm{lk}(22) = C_{11}([23, 13, 10, 11, 19, 20, 21], 4, [5, 16], 17)$, completing successively, we get $\mathrm{lk}(17) = C_{11}([18, 8, 9, 12, 14, 15, 16], 22, [23, 3], 4)$, $\mathrm{lk}(4) = C_{11}([5, 6, 7, 0, 1, 2, 3], 17, [18, 21], 22)$, $\mathrm{lk}(3) = C_{11}([4, 5, 6, 7, 0, 1, 2], 13, [23, 17], 18)$, $\mathrm{lk}(13) = C_{11}([10, 11, 19, 20, 21, 22, 23], 3, [2, 12], 9)$, $\mathrm{lk}(2) = C_{11}([3, 4, 5, 6, 7, 0, 1], 14, [12, 13], 23)$, lk(12)

$= C_{11}([14,15,16,17,18,8,9],10,[13,2],1)$, $\mathrm{lk}(23) = C_{11}([13,10,11,19,20,21,22],16,[17,3],2)$, $\mathrm{lk}(16) = C_{11}([17,18,8,9,12,14,15],6,[5,22],23)$, $\mathrm{lk}(6) = C_{11}([7,0,1,2,3,4,5],16,[15,19],20)$, $\mathrm{lk}(19) = C_{11}([20,21,22,23,13,10,11],14,[15,6],7)$, $\mathrm{lk}(14) = C_{11}([15,16,17,18,8,9,12],2,[1,11],19)$, $\mathrm{lk}(11) = C_{11}([19,20,21,22,23,13,10],0,[1,14],15)$ and $\mathrm{lk}(1) = C_{11}([2,3,4,5,6,7,0],10,[11,14],12)$. This is isomorphic to $K_1(3,4,8,4)$ by the map (0, 7, 6, 5, 4, 3, 2, 1)(8, 20, 14, 10)(9, 19, 12, 11)(13, 18, 21, 15)(16, 23, 17, 22).

Subcase 2.4: When $(m,n) = (5,6)$ then successively we get $\mathrm{lk}(18) = C_{11}([8,9,2,14,15,16,17],6,[5,21],20)$, $\mathrm{lk}(6) = C_{11}([7,0,1,2,3,4,5],18,[17,19],20)$, $\mathrm{lk}(17) = C_{11}([18,8,9,12,14,15,16],11,[19,6],5)$, $\mathrm{lk}(11) = C_{11}([19,20,21,22,23,13,10],0,[1,16],17)$, $\mathrm{lk}(1) = C_{11}([2,3,4,5,6,7,0],10,[11,16],15)$, $\mathrm{lk}(16) = C_{11}([17,18,8,9,12,14,15],2,[1,11],19)$, $\mathrm{lk}(19) = C_{11}([20,21,22,23,13,10,11],16,[17,6],7)$, $\mathrm{lk}(5) = C_{11}([6,7,0,1,2,3,4],22,[21,18],17)$, $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],8,[18,5],4)$ and $\mathrm{lk}(22) = C_{11}([23,13,10,11,19,20,21],5,[4,o],p)$ for some $o,p \in V$. Observe that $(o,p) \in \{(14,12),(14,15)\}$. In case $(o,p) = (14,12)$, we get $C_9(9,10,11,19,20,21,22,23,12) \subseteq \mathrm{lk}(13)$. A contradiction. So $(o,p) = (14,15)$ then $\mathrm{lk}(22) = C_{11}([23,13,10,11,19,20,21],5,[4,14],15)$. This implies $\mathrm{lk}(15) = C_{11}([16,17,18,8,9,12,14],22,[23,2],1)$, completing successively, we get $\mathrm{lk}(2) = C_{11}([3,4,5,6,7,0,1],16,[15,23],13)$, $\mathrm{lk}(13) = C_{11}([10,11,19,20,21,22,23],2,[3,12],9)$, $\mathrm{lk}(12) = C_{11}([14,15,16,17,18,8,9],10,[13,3],4)$, $\mathrm{lk}(4) = C_{11}([5,6,7,0,1,2,3],12,[14,22],21)$ and $\mathrm{lk}(14) = C_{11}([15,16,17,18,8,9,12],3,[4,22],23)$. This is isomorphic to $K_1(3,4,8,4)$ by the map (0, 6)(1, 5)(2, 4)(8, 19, 9, 20)(10, 14, 22, 17)(11, 12, 21, 18)(13, 15, 23, 16).

Case 3: When $(x,y) = (23,22)$ then we get $\mathrm{lk}(8) = C_{11}([9,12,14,15,16,17,18],22,[23,7],0)$, $\mathrm{lk}(23) = C_{11}([13,10,11,19,20,21,22],18,[8,7],6)$, $\mathrm{lk}(7) = C_{11}([0,1,2,3,4,5,6],13,[23,8],9)$. This implies $\mathrm{lk}(13) = C_{11}([10,11,19,20,21,22,23],7,[6,12],9)$, $\mathrm{lk}(6) = C_{11}([7,0,1,2,3,4,5],14,[12,13],23)$, $\mathrm{lk}(12) = C_{11}([14,15,16,17,18,8,9],10,[13,6],5)$ and $\mathrm{lk}(5) = C_{11}([4,3,2,1,0,7,6],12,[14,l],m)$ for some $m,l \in V$. It is easy to see that $(l,m) \in \{(19,20),(20,19),(20,21),(21,20),(21,22)\}$.

When $(l,m) = (21,20)$ then $\mathrm{lk}(5) = C_{11}([6,7,0,1,2,3,4],20,[21,14],12)$. Now considering lk(21) and lk(14) successively we see that lk(22) can not be completed. When $(l,m) = (20,21)$ then $\mathrm{lk}(5) = C_{11}([6,7,0,1,2,3,4],21,[20,14],12)$. This implies $\mathrm{lk}(14) = C_{11}([15,16,17,18,8,9,12],6,[5,20],19)$, $\mathrm{lk}(20) = C_{11}([21,22,23,13,10,11,19],15,[14,5],4)$ and $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],5,[4,n],o)$ for some $n,o \in V$. Observe that $(n,o) \in \{(16,17),(17,16)\}$. If $(n,o) = (16,17)$ then considering lk(21) and lk(22) successively we get $C_9(8,9,12,14,15,16,21,22,18) \subseteq \mathrm{lk}(17)$ and if $(n,o) = (17,16)$ then successively considering lk(21), lk(17) and lk(4) we see easily that lk(22) can not be completed.

This implies $\mathrm{lk}(14) = C_{11}([15,16,17,18,8,9,12],6,[5,19],11)$, completing successively we get $\mathrm{lk}(11) = C_{11}([19,20,21,22,23,13,10],0,[1,15],14)$, $\mathrm{lk}(1) = C_{11}([2,3,4,5,6,7,0],10,[11,15],16)$, $\mathrm{lk}(15) = C_{11}([16,17,18,8,9,12,14],19,[11,1],2)$, $\mathrm{lk}(16) = C_{11}([17,18,8,9,12,14,15],1,[2,21],20)$, $\mathrm{lk}(2) = C_{11}([3,4,5,6,7,0,1],15,[16,21],22)$, $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],17,[16,2],3)$, $\mathrm{lk}(22) = C_{11}([23,13,10,11,19,20,21],2,[3,18],8)$, $\mathrm{lk}(18) = C_{11}([8,9,12,14,15,16,17],4,[3,22],23)$, $\mathrm{lk}(3) = C_{11}([4,5,6,7,0,1,2],21,[22,18],17)$, $\mathrm{lk}(4) = C_{11}([5,6,7,0,1,2,3],18,[17,20],19)$, $\mathrm{lk}(17) = C_{11}([18,8,9,12,14,15,16],21,[20,4],3)$ and lk(20) $= C_{11}([21, 22, 23, 13, 10, 11, 19], 5, [4, 17], 16)$. This is isomorphic to $K_1(3, 4, 8, 4)$ by the map (0, 21, 8, 2, 19, 12, 4, 10, 15, 6, 23, 17)(1, 20, 9, 3, 11, 14, 5, 13, 16, 7, 22, 18).

Subcase 3.2 : When $(l, m) = (20, 19)$ then lk(5) $= C_{11}([6, 7, 0, 1, 2, 3, 4], 19, [20, 14], 12)$. This implies lk(20) $= C_{11}([21, 22, 23, 13, 10, 11, 19], 4, [5, 14], 15)$ and lk(4) $= C_{11}([3, 2, 1, 0, 7, 6, 5], 20, [19, n], o)$ for some $n, o \in V$. Observe that $(n, o) \in \{(16, 15), (16, 17)\}$. If $(n, o) = (16, 17)$ then considering lk(4), lk(16) and lk(11) successively we see that lk(1) can not be completed. On the other hand when $(n, o) = (16, 15)$ then lk(4) $= C_{11}([5, 6, 7, 0, 1, 2, 3], 15, [16, 19], 20)$. This implies lk(15) $= C_{11}([16, 17, 18, 8, 9, 12, 14], 20, [21, 3], 4)$, completing successively we get lk(21) $= C_{11}([22, 23, 13, 10, 11, 19, 20], 14, [15, 3], 2)$, lk(3) $= C_{11}([4, 5, 6, 7, 0, 1, 2], 22, [21, 15], 16)$, lk(16) $= C_{11}([17, 18, 8, 9, 12, 14, 15], 3, [4, 19], 11)$, lk(11) $= C_{11}([19, 20, 21, 22, 23, 13, 10], 0, [1, 17], 16)$, lk(1) $= C_{11}([2, 3, 4, 5, 6, 7, 0], 10, [11, 17], 18)$, lk(18) $= C_{11}([8, 9, 12, 14, 15, 16, 17], 1, [2, 22], 23)$, lk(2) $= C_{11}([3, 4, 5, 6, 7, 0, 1], 17, [18, 22], 21)$ and lk(22) $= C_{11}$ $([23, 13, 10, 11, 19, 20, 21], 3, [2, 18], 8)$. This is isomorphic to $K_1(3, 4, 8, 4)$ by the map (0, 9)(1, 12)(2, 14)(3, 15)(4, 16)(5, 17)(6, 18)(7, 8)(11, 13)(19, 23)(20, 22).

Subcase 3.3 : When $(l, m) = (21, 22)$ then lk(5) $= C_{11}([6, 7, 0, 1, 2, 3, 4], 22, [21, 14], 12)$. This implies lk(22) $= C_{11}([23, 13, 10, 11, 19, 20, 21], 5, [4, 18], 8)$, lk(18) $= C_{11}([8, 9, 12, 14, 15, 16, 17], 3, [4, 22], 23)$ and lk(17) $= C_{11}([16, 15, 14, 12, 9, 8, 18], 4, [3, n], o)$ for some $n, o \in V$. Then we see that $(n, o) \in \{(19,\ 11),\ (19,\ 20),\ (20,\ 19),\ (20,\ 21)\}$. If $(n, o) = (19, 20)$ then successively considering lk(17), lk(19) and lk(3) we see that lk(11) can not be completed. If $(n, o) = (20, 19)$ then successively considering lk(17), lk(20) and lk(3) we see that lk(19) can not be completed. If $(n, o) = (20, 21)$ then successively considering lk(17), lk(21), lk(3) and lk(20) we see that lk(16) can not be completed. If $(o, p) = (19, 11)$ then lk(17) $= C_{11}([18, 8, 9, 12, 14, 15, 16], 11, [19, 3], 4)$, lk(3) $= C_{11}([4, 5, 6, 7, 0, 1, 2], 20, [19, 17], 18)$ and lk (19) $= C_{11}([20, 21, 22, 23, 13, 10, 11], 16, [17, 3], 2)$, completing successively we get lk(11) $= C_{11}([19, 20, 21, 22, 23, 13, 10], 0, [1, 16], 17)$, lk(1) $= C_{11}([2, 3, 4, 5, 6, 7, 0], 10, [11, 16], 15)$, lk (16) $= C_{11}([17, 18, 8, 9, 12, 14, 15], 2, [1, 11], 19)$, lk(2) $= C_{11}([3, 4, 5, 6, 7, 0, 1], 16, [15, 20], 19)$, lk(15) $= C_{11}([16, 17, 18, 8, 9, 12, 14], 21, [20, 2], 1)$, lk(20) $= C_{11}([21, 22, 23, 13, 10, 11, 19], 3, [2, 15], 14)$. This is isomorphic to $K_1(3, 4, 8, 4)$ by the map (0, 18, 20, 4, 14, 13)(1, 17, 21, 5, 12, 10)(2, 16, 22, 6, 9, 11)(3, 15, 23, 7, 8, 19). Thus the Lemma 1.1 is proved.

Proof of Lemma 1.2 : Let M be a SEM of type $(4, 6, 16)$ on the surface of Euler characteristic -1. The notation lk$(i) = C_{20}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, i_{11}, i_{12}, i_{13}, i_{14}, \boldsymbol{i_{15}}, i_{16}, \boldsymbol{i_{17}}, i_{18}, i_{19}, i_{20})$ for the link of i will mean that $[i, i_{15}, i_{16}, i_{17}], [i, i_1, i_{20}, i_{19}, i_{18}, i_{17}]$ and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9\ , i_{10}, i_{11},\ i_{12}, i_{13}, i_{14}, i_{15}]$ form 4-gonal face (face with 4-gonal boundary), 6-gonal face (face with 6-gonal boundary) and 16-gonal face (face with 16-gonal boundary), respectively. Let $|V|$ denote the number of vertices in $V(M)$. If $E(M)$, $Q(M)$, $H(M)$ and $P(M)$ denote the number of edges, number of 4-gonal faces, number of 6-gonal faces and number of 16-gonal faces, respectively, then $E(M) = \frac{3|V|}{2}$, $Q(M) = \frac{|V|}{4}$, $H(M) = \frac{|V|}{6}$ and $P(M) = \frac{|V|}{16}$. By Euler's equation we see if the map exists then $|V| = 48$. For this, let $V = V(M) = \{0,\ 1,\ \ldots,\ 47\}$. Now, we prove the result by exhaustive search for all M.

Assume, $\mathrm{lk}(0) = C_{20}([1,2,3,4,5,6,7,8,9,10,11,12,13,14,\mathbf{15}],16,\mathbf{17},18,19,20)$ then successively we get $\mathrm{lk}(17) = C_{20}([\mathbf{18},21,22,23,24,25,26,27,28,29,30,31,32,33,\mathbf{16}],15,\mathbf{0},1,20,19)$, $\mathrm{lk}(18) = C_{20}([\mathbf{17},16,33,32,31,30,29,28,27,26,25,24,23,22,\mathbf{21}],34,\mathbf{19},20,1,0)$, $\mathrm{lk}(19) = C_{20}([\mathbf{20},35,36,37,38,39,40,41,42,43,44,45,46,47,\mathbf{34}],21,\mathbf{18},17,0,1)$, $\mathrm{lk}(20) = C_{20}([\mathbf{19},34,47,46,45,44,43,42,41,40,39,38,37,36,\mathbf{35}],2,\mathbf{1},0,17,18)$ and $\mathrm{lk}(1) = C_{20}([\mathbf{0},15,14,13,12,11,10,9,8,7,6,5,4,3,\mathbf{2}],35,\mathbf{20},19,18,17)$. This implies $\mathrm{lk}(2) = C_{20}([\mathbf{3},4,5,6,7,8,9,10,11,12,13,14,15,0,\mathbf{1}],20,\mathbf{35},36,d,c)$ for some $c,d \in V$. Then we see that $(c,d) \in \{$(23, 24), (24, 23), (25, 26), (26, 25), (27, 28), (28, 27), (29, 30), (30, 29), (31, 32), (32, 31)$\}$. Observe that, $(29,30) \cong (25,26)$ by the map (0, 19)(1, 20)(2, 35)(3, 36)(4, 37)(5, 38)(6, 39)(7, 40)(8, 41)(9, 42)(10, 43) (11, 44)(12, 45)(13, 46)(14, 47)(15, 34)(16, 21)(17, 18)(22, 33)(23, 32)(24, 31)(25, 30)(26, 29)(27, 28); $(31,32) \cong (23,24)$ by the map (0, 3)(1, 2)(4, 15)(5, 14)(6, 13)(7, 12)(8, 11)(9, 10)(16, 22, 26, 30)(17, 23, 27, 31)(18, 24, 28, 32)(19, 36)(20, 35)(21, 25, 29, 33) (34, 37)(38, 47)(39, 46)(40, 45)(41, 44)(42, 43); $(30,29) \cong (26,25)$ by the map (0, 36)(1, 35)(2, 20)(3, 19)(4, 34)(5, 47) (6, 46)(7, 45)(8, 44)(9, 43)(10, 42)(11, 41)(12, 40)(13, 39)(14, 38)(15, 37)(16, 24, 30, 18, 26, 32, 22, 28)(17, 25, 31, 21, 27, 33, 23, 29). So we have $(c,d) \in \{$(24, 23), (25, 26), (27, 28), (28, 27), (30, 29), (31, 32), (32, 31)$\}$.

If $(c,d) = (24,23)$ then successively considering lk(2), lk(3), lk(23), lk(24), lk(35) and lk(36) we see that lk(21) and lk(22) can not be completed. If $(c,d) = (25,26)$ then successively considering lk(2), lk(3), lk(25), lk(26), lk(35), lk(36) we see that lk(23) and lk(24) can not be completed. If $(c,d) = (28,27)$ then successively considering lk(2), lk(3), lk(27), lk(28), lk(35), lk(36) we see that lk(25) and lk(26) can not be completed. If $(c,d) = (32,31)$ then successively considering lk(2), lk(3), lk(31), lk(32), lk(35), lk(36), we see that lk(16) and lk(33) can not be completed. So we search for $(c,d) \in \{$(27, 28), (30, 29), (31, 32)$\}$.

Case 1 : If $(c,d) = (27,28)$ then constructing successively we get $\mathrm{lk}(2) = C_{20}([\mathbf{3},4,5,6,7,8,9,10,11,12,13,14,15,0,\mathbf{1}],20,\mathbf{35},36,28,27)$, $\mathrm{lk}(3) = C_{20}([\mathbf{2},1,0,15,14,13,12,11,10,9,8,7,6,5,\mathbf{4}],26,\mathbf{27},28,36,35)$, $\mathrm{lk}(27) = C_{20}([\mathbf{28},29,30,31,32,33,16,17,18,21,22,23,24,25,\mathbf{26}],4,\mathbf{3},2,35,36)$, $\mathrm{lk}(28) = C_{20}([\mathbf{27},26,25,24,23,22,21,18,17,16,33,32,31,30,\mathbf{29}],37,\mathbf{36},35,2,3)$, $\mathrm{lk}(35) = C_{20}([\mathbf{36},37,38,39,40,41,42,43,44,45,46,47,34,19,\mathbf{20}],1,\mathbf{2},3,27,28)$, $\mathrm{lk}(36) = C_{20}([\mathbf{35},20,19,34,47,46,45,44,43,42,41,40,39,38,\mathbf{37}],29,\mathbf{28},27,3,2)$ and $\mathrm{lk}(21) = C_{20}([\mathbf{22},23,24,25,26,27,28,29,30,31,32,33,16,17,\mathbf{18}],19,\mathbf{34},47,k,j)$ for some $k,j \in V$. Observe that $(k,j) \in \{$(6, 7), (7, 6), (8, 9), (9, 8), (10, 11), (11, 10), (12, 13), (13, 12)$\}$. If $(k,j) = (7,6)$ then completing lk(6), lk(7), lk(21), lk(22), lk(34), lk(47), lk(4) it is easy to see that lk(24) and lk(25) can not be completed. If $(k,j) \in \{$(8, 9), (11, 10)$\}$ then completing lk(21), lk(22), lk(34), lk(47), lk(k) and lk(j) we see that lk(15) can not be completed. Also, $(8,9) \cong (9,8)$ by the map (0, 17)(1, 18)(2, 21)(3, 22)(4, 23)(5, 24)(6, 25)(7, 26)(8, 27)(9, 28)(10, 29)(11, 30)(12, 31)(13, 32) (14, 33)(15, 16)(34, 34)(36, 47)(37, 46)(38, 45)(39, 44)(40, 43)(41, 42). So we have $(k,j) \in \{$(6, 7), (10, 11), (12, 13), (13, 12)$\}$.

If $(k,j) = (12,13)$ then successively considering lk(12), lk(13), lk(22), lk(21), lk(34), lk(47), lk(23), lk(24), lk(33), lk(16), lk(15), lk(14), lk(4), lk(5), lk(31), lk(32), lk(25) and lk(26) we see that lk(11) can not be completed.

Subcase 1.1 : If $(k, j) = (6, 7)$ then successively we get $\mathrm{lk}(6) = C_{20}([\mathbf{7}, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, \mathbf{5}], 46, \mathbf{47}, 34, 21, 22)$, $\mathrm{lk}(7) = C_{20}([\mathbf{6}, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, \mathbf{8}], 23, \mathbf{22}, 21, 34, 47)$, $\mathrm{lk}(21) = C_{20}([\mathbf{22}, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, \mathbf{18}], 19, \mathbf{34}, 47, 6, 7)$, $\mathrm{lk}(22) = C_{20}([\mathbf{21}, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, \mathbf{23}], 8, \mathbf{7}, 6, 47, 34)$, $\mathrm{lk}\ (34) = C_{20}([\mathbf{47}, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, \mathbf{19}], 18, \mathbf{21}, 22, 7, 6)$, $\mathrm{lk}(47) = C_{20}\ ([\mathbf{34}, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, \mathbf{46}], 5, \mathbf{6}, 7, 22, 21)$, $\mathrm{lk}(4) = C_{20}([\mathbf{5}, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, \mathbf{3}], 27, \mathbf{26}, 25, 45, 46)$, $\mathrm{lk}(5) = C_{20}([\mathbf{4}, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, \mathbf{6}], 47, \mathbf{46}, 45, 25, 26)$, $\mathrm{lk}(45) = C_{20}([\mathbf{46}, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, \mathbf{44}], 24, \mathbf{25}, 26, 4, 5)$, $\mathrm{lk}(46) = C_{20}([\mathbf{45}, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, \mathbf{47}], 6, \mathbf{5}, 4, 26, 25)$, $\mathrm{lk}(25) = C_{20}([\mathbf{26}, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, \mathbf{24}], 44, \mathbf{45}, 46, 5, 4)$, $\mathrm{lk}(26) = C_{20}([\mathbf{25}, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, \mathbf{27}], 3, \mathbf{4}, 5, 46, 45)$, $\mathrm{lk}(8) = C_{20}([\mathbf{9}, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, \mathbf{7}], 22, \mathbf{23}, 24, 44, 43)$, $\mathrm{lk}(9) = C_{20}([\mathbf{8}, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, \mathbf{10}], 42, \mathbf{43}, 44, 24, 23)$, $\mathrm{lk}(23) = C_{20}([\mathbf{24}, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, \mathbf{22}], 7, \mathbf{8}, 9, 43, 44)$, $\mathrm{lk}(24) = C_{20}([\mathbf{23}, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, \mathbf{25}], 45, \mathbf{44}, 43, 9, 8)$, $\mathrm{lk}(44) = C_{20}([\mathbf{43}, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, \mathbf{45}], 25, \mathbf{24}, 23, 8, 9)$ and $\mathrm{lk}(15) = C_{20}([\mathbf{14}, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, \mathbf{0}], 17, \mathbf{16}, 33, m, n)$ for some $m, n \in V$. Observe that $(m, n) \in \{(39,\ 40),\ (40,\ 39),\ (41,\ 42)\}$. In case $(m, n) = (39, 40)$, completing lk(14), lk(15), lk(16), lk(33), lk(39) and lk(40) it is easy to see that lk(30) and lk(31) can not be completed. Also in case $(m, n) = (41, 42)$, completing lk(14), lk(15), lk(16), lk(33), lk(41), lk(42), lk(12), lk(13), lk(24), lk(43), lk(44) we see that lk(22) and lk(23) can not be completed. So $(m, n) = (40, 39)$ then completing successively we get $\mathrm{lk}(15) = C_{20}([\mathbf{14}, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, \mathbf{0}], 17, \mathbf{16}, 33, 40, 39)$, $\mathrm{lk}(16) = C_{20}([\mathbf{33}, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, \mathbf{17}], 0, \mathbf{15}, 14, 39, 40)$, $\mathrm{lk}(14) = C_{20}([\mathbf{15}, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \mathbf{13}], 38, \mathbf{39}, 40, 33, 16)$, $\mathrm{lk}(33) = C_{20}([\mathbf{16}, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, \mathbf{32}], 41, \mathbf{40}, 39, 14, 15)$, $\mathrm{lk}(39) = C_{20}([\mathbf{40}, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, \mathbf{38}], 13, \mathbf{14}, 15, 16, 33)$, $\mathrm{lk}(40) = C_{20}([\mathbf{39}, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, \mathbf{41}], 32, \mathbf{33}, 16, 15, 14)$, $\mathrm{lk}(10) = C_{20}([\mathbf{11}, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, \mathbf{9}], 43, \mathbf{42}, 41, 32, 31)$, $\mathrm{lk}(11) = C_{20}([\mathbf{10}, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, \mathbf{12}], 30, \mathbf{31}, 32, 41, 42)$, $\mathrm{lk}\ (31) = C_{20}([\mathbf{32}, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, \mathbf{30}], 12, \mathbf{11}, 10, 42, 41)$, $\mathrm{lk}(32) = C_{20}([\mathbf{31}, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, \mathbf{33}], 40, \mathbf{41}, 42, 10, 11)$, $\mathrm{lk}(41) = C_{20}([\mathbf{42}, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, \mathbf{40}], 33, \mathbf{32}, 31, 11, 10)$, $\mathrm{lk}(42) = C_{20}([\mathbf{41}, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, \mathbf{43}], 9, \mathbf{10}, 11, 31, 32)$, $\mathrm{lk}(12) = C_{20}([\mathbf{13}, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathbf{11}], 31,\ \mathbf{30}, 29, 37, 38)$, $\mathrm{lk}(13) = C_{20}([\mathbf{12}, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, \mathbf{14}], 39, \mathbf{38}, 37, 29, 30)$, $\mathrm{lk}(29) = C_{20}([\mathbf{30}, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, \mathbf{28}], 36, \mathbf{37}, 38, 13, 12)$, $\mathrm{lk}(30) = C_{20}\ ([\mathbf{29}, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, \mathbf{31}], 11, \mathbf{12}, 13, 38, 37)$, $\mathrm{lk}(37) = C_{20}([\mathbf{38}, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, \mathbf{36}], 28, \mathbf{29}, 30, 12, 13)$, $\mathrm{lk}(38) = C_{20}([\mathbf{37}, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, \mathbf{39}], 14, \mathbf{13}, 12, 30, 29)$, $\mathrm{lk}(43) = C_{20}([\mathbf{44}, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, \mathbf{42}], 10, \mathbf{9}, 8, 23, 24)$. This is isomorphic to $M_1(4, 6, 16)$, as given in Section 2, by the map (0, 9)(1, 8)(2, 7)(3, 6)(4, 5)(10, 15)(11, 14)(12, 13)(16, 39, 31, 42)(17, 38, 30, 43)(18, 37, 29, 44)(19, 25, 34, 24)(20, 26, 47, 23)(21, 36, 28, 45)(22, 35, 27, 46)(32, 41, 33, 40).

Subcase 1.2 : If $(k, j) = (10, 11)$ then constructing successively we get lk(10) $= C_{20}([\mathbf{11}, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, \mathbf{9}], 46, \mathbf{47}, 34, 21, 22)$, lk(11) $= C_{20}([\mathbf{10}, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, \mathbf{12}], 23, \mathbf{22}, 21, 34, 47)$, lk(21) $= C_{20}([\mathbf{22}, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, \mathbf{18}], 19, \mathbf{34}, 47, 10, 11)$, lk(22) $= C_{20}([\mathbf{21}, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, \mathbf{23}], 12, \mathbf{11}, 10, 47, 34)$, lk(34) $= C_{20}([\mathbf{47}, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, \mathbf{19}], 18, \mathbf{21}, 22, 11, 10)$, lk(47) $= C_{20}([\mathbf{34}, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, \mathbf{46}], 9, \mathbf{10}, 11, 22, 21)$ and lk(15) $= C_{20}([\mathbf{14}, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, \mathbf{0}], 17, \mathbf{16}, 33, m, n)$ for some $m, n \in V$. In this case we see that $(m, n) \in \{(39, 40), (40, 39), (41, 42), (42, 41), (43, 44), (44, 43)\}$. If $(m, n) = (39, 40)$ then successively considering lk(14), lk(15), lk(16), lk(33), lk(39), lk(40), lk(13), lk(29), lk(30), lk(37) and lk(38) we see that lk(11) and lk(12) can not be completed. Proceeding similarly for $(m, n) \in \{(40, 39), (41, 42), (42, 41), (43, 44)\}$ it is easy to see that the map does not exist. If $(m, n) = (44, 43)$ then completing successively we get lk(15) $= C_{20}([\mathbf{14}, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, \mathbf{0}], 17, \mathbf{16}, 33, 44, 43)$, lk(16) $= C_{20}([\mathbf{33}, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, \mathbf{17}], 0, \mathbf{15}, 14, 43, 44)$, lk(14) $= C_{20}([\mathbf{15}, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \mathbf{13}], 42, \mathbf{43}, 44, 33, 16)$, lk(33) $= C_{20}([\mathbf{16}, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, \mathbf{32}], 45, \mathbf{44}, 43, 14, 15)$, lk(43) $= C_{20}([\mathbf{44}, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, \mathbf{42}], 13, \mathbf{14}, 15, 16, 33)$, lk(44) $= C_{20}([\mathbf{43}, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, \mathbf{45}], 32, \mathbf{33}, 16, 15, 14)$, lk(4) $= C_{20}([\mathbf{5}, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, \mathbf{3}], 27, \mathbf{26}, 25, 40, 39)$, lk(5) $= C_{20}([\mathbf{4}, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, \mathbf{6}], 38, \mathbf{39}, 40, 25, 26)$, lk (25) $= C_{20}$ $([\mathbf{26}, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, \mathbf{24}], 41, \mathbf{40}, 39, 5, 4)$, lk(26) $= C_{20}([\mathbf{25}, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, \mathbf{27}], 3, \mathbf{4}, 5, 39, 40)$, lk(39) $= C_{20}([\mathbf{40}, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, \mathbf{38}], 6, \mathbf{5}, 4, 26, 25)$, lk(40) $= C_{20}([\mathbf{39}, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, \mathbf{41}], 24, \mathbf{25}, 26, 4, 5)$, lk(6) $= C_{20}([\mathbf{7}, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, \mathbf{5}], 39, \mathbf{38}, 37, 29, 30)$, lk(7) $= C_{20}([\mathbf{6}, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, \mathbf{8}], 31, \mathbf{30}, 29, 37, 38)$, lk(29) $= C_{20}([\mathbf{30}, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, \mathbf{28}], 36, \mathbf{37}, 38, 6, 7)$, lk(30) $= C_{20}([\mathbf{29}, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, \mathbf{31}], 8, \mathbf{7}, 6, 38, 37)$, lk (37) $= C_{20}([\mathbf{38}, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, \mathbf{36}], 28, \mathbf{29}, 30, 7, 6)$, lk(38) $= C_{20}([\mathbf{37}, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, \mathbf{39}], 5, \mathbf{6}, 7, 30, 29)$, lk(8) $= C_{20}([\mathbf{9}, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, \mathbf{7}], 30, \mathbf{31}, 32, 45, 46)$, lk(9) $= C_{20}([\mathbf{8}, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, \mathbf{10}], 47, \mathbf{46}, 45, 32, 31)$, lk(31) $= C_{20}([\mathbf{32}, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, \mathbf{30}], 7, \mathbf{8}, 9, 46, 45)$, lk(32) $= C_{20}([\mathbf{31}, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, \mathbf{33}], 44, \mathbf{45}, 46, 9, 8)$, lk(45) $= C_{20}([\mathbf{46}, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, \mathbf{44}], 33, \mathbf{32}, 31, 8, 9)$, lk(46) $= C_{20}([\mathbf{45}, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, \mathbf{47}], 10, \mathbf{9}, 8, 31, 32)$, lk (12) $= C_{20}([\mathbf{13}, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathbf{11}], 22, \mathbf{23}, 24, 41, 42)$, lk(13) $= C_{20}([\mathbf{12}, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, \mathbf{14}], 43, \mathbf{42}, 41, 24, 23)$, lk(23) $= C_{20}([\mathbf{24}, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, \mathbf{22}], 11, \mathbf{12}, 13, 42, 41)$, lk (24) $= C_{20}([\mathbf{23}, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, \mathbf{25}], 40, \mathbf{41}, 42, 13, 12)$, lk(41) $= C_{20}$ $([\mathbf{42}, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, \mathbf{40}], 25, \mathbf{24}, 23, 12, 13)$, lk(42) $= C_{20}([\mathbf{41}, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44,$

43], 14, **13**, 12, 23, 24). This map is isomorphic to $M_2(4,6,16)$ as given in Section 2, by the map (0, 7)(1, 6)(2, 5)(3, 4)(8, 15)(9, 14)(10, 13)(11, 12)(16, 31) (17, 30)(18, 29)(19, 45, 41, 37)(20, 46, 42, 38)(21, 28)(22, 27)(23, 26)(24, 25)(32, 33)(34, 44, 40, 36)(35, 47, 43, 39).

Subcase 1.3: If $(k, j) = (13, 12)$ then successively we get lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 23, **22**, 21, 34, 47), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**],46, **47**, 34, 21, 22), lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 13, 12), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 11, **12**, 13, 47, 34), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**], 18, **21**, 22, 12, 13), lk(47) = C_{20}([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 14, **13**, 12, 22, 21), lk(33) = C_{20}([**16**, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, **32**], 44, **45**, 46, 14, 15), lk(14) = C_{20}([**15**, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, **13**], 47, **46**, 45, 33, 16), lk(15) = C_{20}([**14**, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, **0**], 17, **16**, 33, 45, 46), lk(16) = C_{20}([**33**, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, **17**], 0, **15**, 14, 46, 45), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**],32, **33**, 16, 15, 14), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 13, **14**, 15, 16, 33) and lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 33, **32**, 31, m, n) for some $m, n \in V$. In this case, $(m, n) \in$ {(6, 7), (7, 6), (8, 9), (9, 8)}. If $(m, n) = (6, 7)$ then successively considering lk(6), lk(7), lk(31), lk(32), lk(43), lk(44), lk(4), lk(5), we see 25 29 as an edge and a non-edge both. If $(m, n) = (9, 8)$ then successively considering lk(8), lk(9), lk(31), lk(32), lk(43), lk(44), lk(10), lk(11) we see 24 29 as an edge and a non-edge both. So we have $(m, n) \in$ {(7, 6), (8, 9)}.

If $(m, n) = (7, 6)$ then completing successively we get lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 33, **32**, 31, 7, 6), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 42, **43**, 44, 32, 31), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 30, **31**, 32, 44, 43), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 8, **7**, 6, 43, 44), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**], 45, **44**, 43, 6, 7), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 5, **6**, 7, 31, 32), lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 27, **26**, 25, 41, 42), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 43, **42**, 41, 25, 26), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**],40, **41**, 42, 5, 4), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 3, **4**, 5, 42, 41), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 24, **25**, 26, 4, 5), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 6, **5**, 4, 26, 25), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 31, **30**, 29, 37, 38), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 39, **38**, 37, 29, 30), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 36, **37**, 38, 9, 8), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 7, **8**, 9, 38, 37), lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 28, **29**, 30, 8, 9), lk(38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**], 10, **9**, 8, 30, 29), lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 38, **39**, 40, 24, 23), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 22, **23**, 24, 40, 39), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 12, **11**, 10, 39, 40), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 41, **40**, 39, 10, 11), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**],9, **10**, 11, 23, 24), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 25, **24**, 23, 11, 10). This is isomorphic to $M_2(4, 6, 16)$ by the map (0, 1)(2, 15)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)(16, 35) (17, 20)(18, 19)(21, 34)(22, 47)(23, 46)(24, 45)(25, 44)(26, 43)(27, 42)(28, 41)(29, 40)(30, 39)(31, 38)(32, 37)(33, 36).

If $(m, n) = (8, 9)$ then completing successively we get lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 33, **32**, 31, 8, 9), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 10, **9**, 8, 31, 32), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 7, **8**, 9, 43, 44), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**],45, **44**, 43, 9, 8), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 30, **31**, 32, 44, 43), lk (9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 42, **43**, 44, 32, 31),lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 27, **26**, 25, 40, 39), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 38, **39**, 40, 25, 26), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 41, **40**, 39, 5, 4), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 3, **4**, 5, 39, 40), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**], 6, **5**, 4, 26, 25), lk (40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 24, **25**, 26, 4, 5), lk(6) = C_{20} ([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 39, **38**, 37, 29, 30), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 31, **30**, 29, 37, 38), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 36, **37**, 38, 6, 7), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 8, **7**, 6, 38, 37), lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 28, **29**, 30, 7, 6), lk(38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**], 5, **6**, 7, 30, 29), lk (10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 43, **42**, 41, 24, 23), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 22, **23**, 24, 41, 42), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 12, **11**, 10, 42, 41), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 40, **41**, 42, 10, 11), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 25, **24**, 23, 11, 10), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 9, **10**, 11, 23, 24). This is isomorphic to $M_1(4, 6, 16)$ by the map (0, 1)(2, 15)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)(16, 35)(17, 20)(18, 19)(21, 34)(22, 47)(23, 46)(24, 45)(25, 44)(26, 43)(27, 42)(28, 41)(29, 40)(30, 39)(31, 38)(32, 37)(33, 36). This completes the search for $(c, d) = (27, 28)$.

Case 2 : If $(c, d) = (30, 29)$ then constructing successively we get lk(2) = C_{20}([**3**, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, **1**], 20, **35**, 36, 29, 30), lk(3) = C_{20}([**2**, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, **4**], 31, **30**, 29, 36, 35), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 37, **36**, 35, 2, 3), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 4, **3**, 2, 35, 36), lk(35) = C_{20}([**36**, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, **20**], 1, **2**, 3, 30, 29), lk(36) = C_{20}([**35**, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, 39, 38, **37**], 28, **29**, 30, 3, 2) and lk(31) = C_{20}([**32**, 33,

16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 3, **4**, 5, k, j) for some $k, j \in V$. In this case, $(k, j) \in \{(39, 40), (40, 39), (41, 42), (42, 41), (43, 44), (44, 43), (45, 46), (46, 45)\}$. If $(k, j) = (40, 39)$ then successively considering lk(4), lk(5), lk(31), lk(32), lk(39), lk(40), lk(33), lk(27), lk(28), lk(37), lk(38), it is easy to see that lk(16) and lk(17) can not be completed. Now, proceeding similarly for $(k, j) \in \{(41, 42), (44, 43), (45, 46)\}$, we see that the map does not exist. Also, $(42, 41) \cong (46, 45)$ by the map (0, 5)(1, 4)(2, 3)(6, 15)(7, 14)(8, 13) (9, 12)(10, 11)(16, 47)(17, 46)(18, 45)(19, 32)(20, 31)(21, 44)(22, 43)(23, 42)(24, 41)(25, 40)(26, 39)(27, 38)(28, 37)(29, 36)(30, 35). So, we search the map for $(k, j) \in \{(39, 40), (42, 41), (43, 44)\}$.
Subcase 2.1 : If $(k, j) = (39, 40)$ then constructing successively we get lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 30, **31**, 32, 40, 39), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**],38, **39**, 40, 32, 31), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**], 6, **5**, 4, 31, 32), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 33, **32**, 31, 4, 5), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 3, **4**, 5, 30, 40), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**], 41, **40**, 39, 5, 4), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 39, **38**, 37, 28, 27), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 26, **27**, 28, 37, 38), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 8, **7**, 6, 38, 37), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 36, **37**, 38, 6, 7), lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 29, **28**, 27, 7, 6), lk (38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**], 5, **6**, 7, 27, 28), lk(14) = C_{20}([**15**, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, **13**], 43, **42**, 41, 33, 16), lk(15) = C_{20}([**14**, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, **0**], 17, **16**, 33, 41, 42), lk(16) = C_{20}([**33**, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, **17**], 0, **15**, 14, 42, 41), lk(33) = C_{20}([**16**, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, **32**],40, **41**, 42, 14, 15), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 32, **33**, 16, 15, 14), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 13, **14**, 15, 16, 33) and lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, m, n) for some $m, n \in V$. In this case, $(m, n) \in \{(10, 11), (11, 10)\}$. If $(m, n) = (11, 10)$ then successively considering lk(10), lk(11), lk(21), lk(22), lk(34), lk(47), lk(12), lk(13), we see that lk(44) can not be completed. So $(m, n) = (10, 11)$.

Then completing successively we get lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 10, 11), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 12, **11**, 10, 47, 34), lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 46, **47**, 34, 21, 22), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 23, **22**, 21, 34, 47), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**], 18, **21**, 22, 11, 10), lk(47) = C_{20}([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 9, **10**, 11, 22, 21), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 27, **26**, 25, 45, 46), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 47, **46**, 45, 25, 26), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 44, **45**, 46, 9, 8), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 7, **8**, 9, 46, 45), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 24, **25**, 26, 8, 9), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 10, **9**, 8, 26, 25), lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 22, **23**, 24, 44, 43), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 42, **43**, 44, 24, 23), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 11, **12**, 13, 43, 44), lk (24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 45, **44**, 43, 13, 12), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 14, **13**, 12, 23, 24), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 25, **24**, 23, 12, 13). This is isomorphic to $M_1(4,6,16)$ by the map (0, 7)(1, 6)(2, 5)(3, 4)(8, 15)(9, 14)(10, 13)(11, 12)(16, 26)(17, 27)(18, 28)(19, 45, 41, 37)(20, 46, 42, 38)(21, 29)(22, 30)(23, 31)(24, 32)(25, 33)(34, 44, 40, 36)(35, 47, 43, 39).

Subcase 2.2 : If $(k,j) = (42,41)$ then successively we get lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 30, **31**, 32, 41, 42), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 43, **42**, 41, 32, 31), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 33, **32**, 31, 4, 5), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 6, **5**, 4, 31, 32), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 3, **4**, 5, 42, 41), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**], 40, **41**, 42, 5, 4), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 39, **38**, 37, 28, 27), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 11, **12**, 13, 38, 37), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 36, **37**, 38, 13, 12), lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 29, **28**, 27, 12, 13), lk(38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**],14, **13**, 12, 27, 28) and lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, m, n) for some $m, n \in V$. In this case, $(m,n) \in \{(8, 9), (9, 8)\}$. If $(m,n) = (8,9)$ then successively considering lk(8), lk(9), lk(21), lk(22), lk(34), lk(47), we see that lk(24) and lk(25) can not be completed. If $(m,n) = (9,8)$ then completing successively we get lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 9, 8), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 7, **8**, 9, 47, 34), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 23, **22**, 21, 34, 47), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 46, **47**, 34, 21, 22), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**], 18, **21**, 22, 8, 9), lk(47) = C_{20}([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 10, **9**, 8, 22, 21), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 42, **43**, 44, 24, 23), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 22, **23**, 24, 44, 43), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 8, **7**, 6, 43, 44), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 45, **44**, 43, 6, 7), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 5, **6**, 7, 23, 24), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 25, **24**, 23, 7, 6), lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 47, **46**, 45, 25, 26), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 27, **26**, 25, 45, 46), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 44, **45**, 46, 10, 11), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 12, **11**, 10, 46, 45), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 24, **25**, 26, 11, 10), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 9, **10**, 11, 26, 25), lk(14) = C_{20}([**15**, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, **13**], 38, **39**, 40, 33, 16), lk(15) = C_{20}([**14**, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, **0**], 17, **16**, 33, 40, 39), lk(16) = C_{20}([**33**, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, **17**], 0, **15**, 14, 39, 40), lk(33) = C_{20}([**16**, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, **32**], 41, **40**, 39, 14, 15), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**], 13, **14**, 15, 16, 33). This is isomorphic to $M_2(4,6,16)$ by the map (0, 11)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)(12, 15)(13, 14)(16, 27)(17, 26)(18, 25)(19, 37, 41, 45)(20, 38, 42, 46)(21, 24)(22, 23)(28, 33)(29, 32)(30, 31)(34, 36, 40, 44)(35, 39, 43, 47).

Subcase 2.3: If $(k,j) = (43,44)$ then constructing successively we get lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 30, **31**, 32, 44, 43), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**],42, **43**, 44, 32, 31), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 6, **5**, 4, 31, 32), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 33, **32**, 31, 4, 5), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 3, **4**, 5, 43, 44), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**], 45, **44**, 43, 5, 4), lk(14) = C_{20}([**15**, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, **13**], 47, **46**, 45, 33, 16), lk(15) = C_{20}([**14**, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, **0**], 17, **16**, 33, 45, 46), lk(16) = C_{20}([**33**, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, **17**], 0, **15**, 14, 46, 45), lk(33) = C_{20}([**16**, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, **32**], 44, **45**, 46, 14, 15), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 32, **33**, 16, 15, 14), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 13, **14**, 15, 16, 33), lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 23, **22**, 21, 34, 47), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 46, **47**, 34, 21, 22), lk(47) = C_{20}([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 14, **13**, 12, 22, 21), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**], 18, **21**, 22, 12, 13), lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 13, 12), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 11, **12**, 13, 47, 34) and lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 29, **28**, 27, m, n) for some $m, n \in V$. Observe that $(m,n) \in \{(8, 9), (9, 8)\}$. In case $(m,n) = (9,8)$, completing lk(8), lk(9), lk(27), lk(28), lk(37), lk(38), we see easily that lk(24) and lk(25) can not be completed. On the other hand when $(m,n) = (8,9)$ then completing successively we get lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 29, **28**, 27, 8, 9), lk(38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**], 10, **9**, 8, 27, 28), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 26, **27**, 28, 37, 38), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 39, **38**, 37, 28, 27), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 7, **8**, 9, 38, 37), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 36, **37**, 38, 9, 8), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 43, **42**, 41, 25, 26), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 27, **26**, 25, 41, 42), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 24, **25**, 26, 7, 6), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 5, **6**, 7, 26, 25), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 40, **41**, 42, 6, 7), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 8, **7**, 6, 42, 41), lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 38, **39**, 40, 24, 23), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 22, **23**, 24, 40, 39), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 12, **11**, 10, 39, 40), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 41, **40**, 39, 10, 11), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**],9, **10**, 11, 23, 24), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 25, **24**, 23, 11, 10). This is isomorphic to $M_1(4, 6, 16)$ by the map (0, 11)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)(12, 15)(13, 14)(16, 30, 26, 22)(17, 31, 27, 23)(18, 32, 28, 24)(19, 40)(20, 39)(21, 33, 29, 25)(34, 41)(35, 38)(36, 37)(42, 47)(43, 46)(44, 45). This completes the search for $(c, d) = (30, 29)$.

Case 3: If $(c, d) = (31, 32)$ then successively we get lk(3) = C_{20}([**2**, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, **4**], 30, **31**, 32, 36, 35), lk(35) = C_{20}([**36**, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, **20**], 1, **2**, 3, 31, 32), lk(36) = C_{20}([**35**, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, 39, 38, **37**], 33, **32**, 31, 3, 2), lk(31) = C_{20}([**32**, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, **30**], 4, **3**, 2, 35, 36), lk(32) = C_{20}([**31**, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, **33**], 37, **36**, 35, 2, 3), lk(14) = C_{20}([**15**, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, **13**], 39, **38**, 37, 33, 16), lk(15) = C_{20}([**14**, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, **0**], 17, **16**, 33, 37, 38), lk(16) = C_{20}([**33**, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 18, **17**], 0, **15**, 14, 38, 37), lk(33) = C_{20}([**16**, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, **32**], 36, **37**, 38, 14, 15), lk(37) = C_{20}([**38**, 39, 40, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, **36**], 32, **33**, 16, 15, 14), lk(38) = C_{20}([**37**, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, 41, 40, **39**], 13, **14**, 15, 16, 33) and lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, k, j) for some $k, j \in V$. Then we see that $(k, j) \in \{(6, 7), (8, 9), (9, 8), (10, 11)\}$. In case $(k, j) = (8, 9)$, considering lk(8), lk(9), lk(22), lk(21), lk(34) and lk(47) successively we see that lk(7) can not be completed. Also, $(6, 7) \cong (10, 11)$ by the map (0, 1)(2, 15)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)(16, 35)(17, 20)(18, 19)(21, 34)(22, 47)(23, 46)(24, 45)(25, 44)(26, 43)(27, 42)(28, 41)(29, 40)(30, 39)(31, 38)(32, 37)(33, 36). So we search for $(k, j) \in \{(6, 7), (9, 8)\}$.

Subcase 3.1: If $(k, j) = (6, 7)$ then constructing successively we get lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 46, **47**, 34, 21, 22), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**],23, **22**, 21, 34, 47), lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 6, 7), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 8, **7**, 6, 47, 34), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**],18, **21**, 22, 7, 6), lk(47) = C_{20}([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 5, **6**, 7, 22, 21), lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 31, **30**, 29, 45, 46), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 47, **46**, 45, 29, 30), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 28, **29**, 30, 4, 5), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 6, **5**, 4, 30, 29), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 44, **45**, 46, 5, 4), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 3, **4**, 5, 46, 45) and lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 38, **39**, 40, m, n) for some $m, n \in V$. Observe that $(m, n) \in$ {(25, 26), (26, 25)}. In case $(m, n) = (26, 25)$, completing lk(12), lk(13), lk(25), lk(26), lk(39), lk(40), lk(23), lk(24), it is easy to see that lk(9) and lk(10) can not be completed. On the other hand when $(m, n) = (25, 26)$ then completing successively we get lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 27, **26**, 25, 40, 39), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 38, **39**, 40, 25, 26), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 41, **40**, 39, 13, 12), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 11, **12**, 13, 39, 40), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**],14, **13**, 12, 26, 25), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 24, **25**, 26, 12, 13), lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 22, **23**, 24, 41, 42), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**], 43, **42**, 41, 24, 23), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 7, **8**, 9, 42, 41), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 40, **41**, 42, 9, 8), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 25, **24**, 23, 8, 9), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 10, **9**, 8, 23, 24), lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**],42, **43**, 44, 28, 27), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 26, **27**, 28, 44, 43), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 12, **11**, 10, 43, 44), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 45, **44**, 43, 10, 11), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 9, **10**, 11, 27, 28), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 29, **28**, 27, 11, 10). This is isomorphic to $M_1(4, 6, 16)$ by the map (0, 13)(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(14, 15)(16, 42, 22, 46, 26, 20, 30, 38)(17, 43, 23, 47, 27, 35, 31, 39)(18, 44, 24, 34, 28, 36, 32, 40)(19, 29, 37, 33, 41, 21, 45, 25).

Subcase 3.2 : If $(k, j) = (9, 8)$ then constructing successively we get lk(8) = C_{20}([**9**, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, **7**], 23, **22**, 21, 34, 47), lk(9) = C_{20}([**8**, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, **10**],46, **47**, 34, 21, 22), lk(22) = C_{20}([**21**, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, **23**], 7, **8**, 9, 47, 34), lk(21) = C_{20}([**22**, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, **18**], 19, **34**, 47, 9, 8), lk(34) = C_{20}([**47**, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, **19**], 18, **21**, 22, 8, 9), lk(47) = C_{20} ([**34**, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, **46**], 10, **9**, 8, 22, 21) and lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 47, **46**, 45, m, n) for some $m, n \in V$. It is easy to see that $(m, n) \in \{(25, 26), (28, 27)\}$.

If $(m, n) = (25, 26)$ then completing successively we get lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 47, **46**, 45, 25, 26), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**],27, **26**, 25, 45, 46), lk(25) = C_{20}([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 44, **45**, 46, 10, 11), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 12, **11**, 10, 46, 45), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 24, **25**, 26, 11, 10), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 9, **10**, 11, 26, 25), lk(4) = C_{20} ([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 31, **30**, 29, 41, 42), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 43,

42, 41, 29, 30), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 40, **41**, 42, 5, 4), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 3, **4**, 5, 42, 41), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 28, **29**, 30, 4, 5), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 6, **5**, 4, 30, 29), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 42, **43**, 44, 24, 23), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 22, **23**, 24, 44, 43), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 8, **7**, 6, 43, 44), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 45, **44**, 43, 6, 7), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 5, **6**, 7, 23, 24), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 25, **24**, 23, 7, 6), lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 26, **27**, 28, 40, 39), lk(13) = C_{20} ([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 38, **39**, 40, 28, 27), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 11, **12**, 13, 39, 40), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 41, **40**, 39, 13, 12), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**], 14, **13**, 12, 27, 28), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 29, **28**, 27, 12, 13). This is isomorphic to $M_1(4, 6, 16)$ by the map (0, 20, 18)(1, 19, 17)(2, 34, 16)(3, 47, 33)(4, 46, 32)(5, 45, 31)(6, 44, 30)(7, 43, 29)(8, 42, 28)(9, 41, 27)(10, 40, 26)(11, 39, 25)(12, 38, 24)(13, 37, 23)(14, 36, 22)(15, 35, 21).

On the other hand when, $(m, n) = (28, 27)$ then completing successively we get lk(10) = C_{20}([**11**, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, **9**], 47, **46**, 45, 28, 27), lk(11) = C_{20}([**10**, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, 14, 13, **12**], 26, **27**, 28, 45, 46), lk(27) = C_{20}([**28**, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, **26**], 12, **11**, 10, 46, 45), lk(28) = C_{20}([**27**, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, **29**], 44, **45**, 46, 10, 11), lk(45) = C_{20}([**46**, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, 42, 43, **44**], 29, **28**, 27, 11, 10), lk(46) = C_{20}([**45**, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, **47**], 9, **10**, 11, 27, 28), lk(4) = C_{20}([**5**, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, **3**], 31, **30**, 29, 44, 43), lk(5) = C_{20}([**4**, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, 8, 7, **6**], 42, **43**, 44, 29, 30), lk(29) = C_{20}([**30**, 31, 32, 33, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, **28**], 45, **44**, 43, 5, 4), lk(30) = C_{20}([**29**, 28, 27, 26, 25, 24, 23, 22, 21, 18, 17, 16, 33, 32, **31**], 3, **4**, 5, 43, 44), lk(43) = C_{20}([**44**, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, 40, 41, **42**], 6, **5**, 4, 30, 29), lk(44) = C_{20}([**43**, 42, 41, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, **45**], 28, **29**, 30, 4, 5), lk(6) = C_{20}([**7**, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, **5**], 43, **42**, 41, 24, 23), lk(7) = C_{20}([**6**, 5, 4, 3, 2, 1, 0, 15, 14, 13, 12, 11, 10, 9, **8**], 22, **23**, 24, 41, 42), lk(23) = C_{20}([**24**, 25, 26, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, **22**], 8, **7**, 6, 42, 41), lk(24) = C_{20}([**23**, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, 27, 26, **25**], 40, **41**, 42, 6, 7), lk(41) = C_{20}([**42**, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, 38, 39, **40**], 25, **24**, 23, 7, 6), lk(42) = C_{20}([**41**, 40, 39, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, **43**], 5, **6**, 7, 23, 24), lk(12) = C_{20}([**13**, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, **11**], 27, **26**, 25, 40, 39), lk(13) = C_{20}([**12**, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 15, **14**], 38, **39**, 40, 25, 26), lk(25) = C_{20} ([**26**, 27, 28, 29, 30, 31, 32, 33, 16, 17, 18, 21, 22, 23, **24**], 41, **40**, 39, 13, 12), lk(26) = C_{20}([**25**, 24, 23, 22, 21, 18, 17, 16, 33, 32, 31, 30, 29, 28, **27**], 11, **12**, 13, 39, 40), lk(39) = C_{20}([**40**, 41, 42, 43, 44, 45, 46, 47, 34, 19, 20, 35, 36, 37, **38**],14, **13**, 12, 26, 25), lk(40) = C_{20}([**39**, 38, 37, 36, 35, 20, 19, 34, 47, 46, 45, 44, 43, 42, **41**], 24, **25**, 26, 12, 13). This is isomorphic to $M_2(4,6,16)$ by the map (0, 3)(1, 2)(4, 15)(5, 14)(6, 13)(7, 12)(8, 11)(9, 10)(16, 22, 26, 30)(17, 23, 27, 31)(18, 24, 28, 32)(19, 36)(20, 35)(21, 25, 29, 33)(34, 37)(38, 47)(39, 46)(40, 45)(41, 44)(42, 43). This completes the search for $(c,d) = (31,32)$ and thus the Lemma 1.2 is proved. $\square$

Proof of Lemma 1.3 : Let N be a SEM of type $(6^2, 8)$ on the surface of Euler characteristic -1. The notation $\text{lk}(i) = C_{14}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, \boldsymbol{i_7}], i_8, i_9, i_{10}, \boldsymbol{i_{11}}, i_{12}, i_{13}, i_{14})$ for the link of i will mean that $[i, i_7, i_8, i_9, i_{10}, i_{11}], [i, i_1, i_{14}, i_{13}, i_{12}, i_{11}]$ form hexagonal faces and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7]$ forms octagonal face. If $|V|$, $E(N)$, $H(N)$ and $O(N)$ denote number vertices, number of edges, number of hexagonal faces and number of octagonal faces in the map N, respectively, then $E(N) = \frac{3|V|}{2}$, $H(N) = \frac{2|V|}{6}$ and $O(N) = \frac{|V|}{8}$. Using Euler's equation we see that if the map exists then $|V| = 24$. Let $V = V(M) = \{0, 1, \ldots, 23\}$. Now, we prove the proposition by exhaustive search for all N.

Assume that, $\text{lk}(0) = C_{14}([\mathbf{1}, 2, 3, 4, 5, 6, \mathbf{7}], 8, 9, 10, \mathbf{11}, 12, 13, 14)$. This implies $\text{lk}(11) = C_{14}([\mathbf{12}, 19, 18, 17, 16, 15, \mathbf{10}], 9, 8, 7, \mathbf{0}, 1, 14, 13)$ and $\text{lk}(8) = C_{14}([\mathbf{9}, f, e, d, c, b, \boldsymbol{a}], g, h, 6, \mathbf{7}, 0, 11, 10)$ for some $a, b, c, d, e, f, g, h \in V$. Then we get the partial picture of the map as shown in Figure II. Let $V(O_i)$, for $i = 1, 2, 3$, denote the vertex set of octagonal face O_i then we see that $V(O_1) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $V(O_2) = \{10, 11, 12, 15, 16, 17, 18, 19\}$ and $V(O_3) = \{8, 9, 13, 14, 20, 21, 22, 23\}$. In this case we observe that $a \in \{13, 14, 20\}$. If $a = 14$ then completing successively we get $b = 13$, $c = 20$, $d = 21$, $e = 22$ and $f = 23$. This implies $g = 1$. This contradicts the fact that $g \in V(O_2)$, as $1 \in O_1$. So $a \neq 14$. So, $a = 13$ or 20.

Case 1 : If $a = 13$ then successively we get $b = 14$, $c = 20$, $d = 21$, $e = 22$, $f = 23$, $g = 12$ and $h = 19$. This implies $\text{lk}(8) = C_{14}([\mathbf{9}, 23, 22, 21, 20, 14, \mathbf{13}], 12, 19, 6, \mathbf{7}, 0, 11, 10)$, $\text{lk}(7) = C_{14}([\mathbf{0}, 1, 2, 3, 4, 5, \mathbf{6}], 19, 12, 13, \mathbf{8}, 9, 10, 11)$, $\text{lk}(12) = C_{14}([\mathbf{19}, 18, 17, 16, 15, 10, \mathbf{11}], 0, 1, 14, \mathbf{13}, 8, 7, 6)$, $\text{lk}(13) = C_{14}([\mathbf{8}, 9, 23, 22, 21, 20, \mathbf{14}], 1, 0, 11, \mathbf{12}, 19, 6, 7)$ and $\text{lk}(19) = C_{14}([\mathbf{18}, 17, 16, 15, 10, 11, \mathbf{12}], 13, 8, 7, \mathbf{6}, 5, j, i)$ for some $i, j \in V(O_3)$. Then we see that $(j, i) \in \{(20, 21), (21, 20), (21, 22), (23, 22)\}$. Observe that $(23, 22) \cong (21, 20)$ by the map (0, 11)(1, 10)(2, 15)(3, 16)(4, 17)(5, 18)(6, 19)(7, 12)(8, 13)(9, 14)(20, 23)(21, 22), so we search for $(j, i) \in \{(20, 21), (21, 20), (21, 22)\}$.

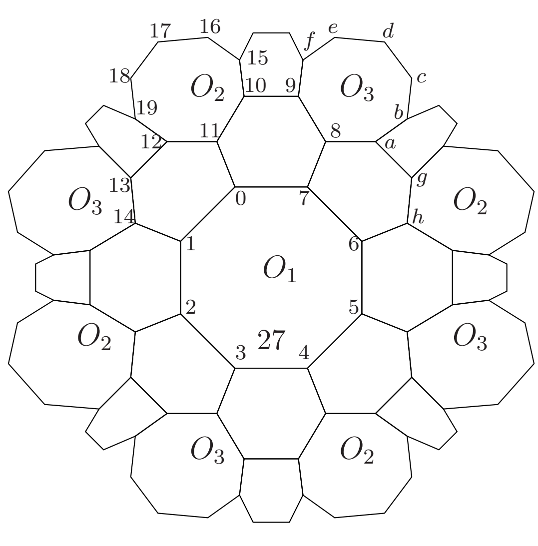


Figure II: Semi-equivelar map N of type $(6^2, 8)$

If $(j,i) = (20,21)$ then lk(19) = C_{14}([**18**, 17, 16, 15, 10, 11, **12**], 13, 8, 7, **6**, 5, 20, 21) and lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 21, **20**, 14, k, l) for some $k, l \in V(O_2)$. This implies deg(14) > 3, a contradiction. So $(j,i) \neq (20,21)$.

Subcase 2.1 : If $(j,i) = (21,20)$ then lk(19) = C_{14}([**18**, 17, 16, 15, 10, 11, **12**], 13, 8, 7, **6**, 5, 21, 20), lk(6) = C_{14}([**7**, 0, 1, 2, 3, 4, **5**], 21, 20, 18, **19**, 12, 13, 8) and lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 20, **21**, 22, k, l) for some $k, l \in V(O_2)$. Observe that $(l,k) \in \{$(15, 16), (16, 15), (16, 17), (17, 16)$\}$. If $(l,k) = (17,16)$ then successively considering lk(5) and lk(4) we get deg(14) > 3. A contradiction. If $(l,k) = (16,15)$ then $C_{13}(4,5,21,22,23,9,10,11,12,19,18,17,16) \subseteq$ lk(15). A contradiction. If $(l,k) = (16,17)$ then considering lk(5) and we see that lk(15) and lk(16) can not be completed. If $(l,k) = (15,16)$ then successively we get lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 20, **21**, 22, 16, 15), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **4**, 5, 21, 22), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 4), lk(9) = C_{14}([**8**, 13, 14, 20, 21, 22, **23**], 3, 4, 15, **10**, 11, 0, 7) and lk(23) = C_{14}([**22**, 21, 20, 14, 13, 8, **9**], 10, 15, 4, **3**, 2, n, m) for some $m, n \in V(O_2)$. Observe that $n = 17$ and $m = 16$. This implies lk(23) = C_{14}([**22**, 21, 20, 14, 13, 8, **9**], 10, 15, 4, **3**, 2, 17, 16), completing successively we get lk(22) = C_{14}([**23**, 9, 8, 13, 14, 20, **21**], 5, 4, 16, **15**, 17, 2, 3), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 2, 3, 23, **22**, 21, 5, 4), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 14, 1, **2**, 3, 23, 22), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 5, 21, **20**, 14, 1, 2), lk(1) = C_{14}([**2**, 3, 4, 5, 6, 7, **0**], 11, 12, 13, **14**, 20, 18, 17), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 20, 18, **17**, 16, 22, 23), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 17, 16, 22, **23**, 9, 10, 15), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 9, 10, **15**, 16, 22, 21), lk(14) = C_{14}([**20**, 21, 22, 23, 9, 8, **13**], 12, 11, 0, **1**, 2, 17, 18) and lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**],6, 7, 8, **13**, 14, 1, 0). This is $N_1(6^2, 8)$ as given in Section 2.

Subcase 2.2 : If $(j,i) = (21,22)$ then lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 13, 12, **19**, 18, 22, 21). This implies lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 22, **21**, 20, l, k) for some $k, l \in V$. Then we see that $(k,l) \in \{$(15, 16), (16, 17), (17, 16)$\}$. In case $(k,l) = (16,17)$, completing lk(5) and lk(20) we see that lk(15) can not be completed. So we search for $(k,l) \in \{$(15, 16), (17, 16)$\}$.

Subcase 2.2.1 : If $(k,l) = (15,16)$ then lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 22, **21**, 20, 16, 15), completing successively we get lk(21) = C_{14}([**20**, 14, 13, 8, 9, 23, **22**], 18, 19, 6, **5**, 4, 15, 16), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 20, 16, **17**, 18, 22, 23), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**],22, 23, 3, **2**, 1, 14, 20), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 17, 18, 22, **23**, 9, 10, 15), lk(23) = C_{14}([**9**, 8, 13, 14, 20, 21, **22**], 18, 17, 2, **3**, 4, 15, 10), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 9, 10, **15**, 16, 20, 21), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **4**, 5, 21, 20), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 22, **21**, 20, 16, 15), lk(9) = C_{14}([**8**, 13, 14, 20, 21, 22, **23**], 3, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 4), lk(14) = C_{14}([**20**, 21, 22, 23, 9, 8, **13**], 12, 11, 0, **1**, 2, 17, 16), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 2, 1, 14, **20**, 21, 5, 4), lk(20) = C_{14}([**21**, 22, 23, 9, 8, 13, **14**], 1, 2, 17, **16**, 15, 4, 5), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 5, 21, **22**, 23, 3, 2), lk(22) = C_{14}([**23**, 9, 8, 13, 14, 20, **21**], 5, 6, 19, **18**, 17, 2, 3). This is isomorphic to

$N_2(6^2, 8)$ as given in Section 2, by the map (0, 5, 2, 7, 4, 1, 6, 3)(8, 19, 23, 11, 21, 15, 14, 17)(9, 12, 22, 10, 13, 18)(16, 20).

Subcase 2.2.2 : If $(k, l) = (17, 16)$ then lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 18, 22, **21**, 20, 16, 17), lk(21) = C_{14}([**20**, 14, 13, 8, 9, 23, **22**], 18, 19, 6, **5**, 4, 17, 16), completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 17, 16, 20, **14**, 13, 12, 11), lk(14) = C_{14}([**20**, 21, 22, 23, 9, 8, **13**], 12, 11, 0, **1**, 2, 15, 16), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 20, 16, **15**, 10, 9, 23), lk(15) = C_{14} ([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **2**, 1, 14, 20), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 15, 10, 9, **23**, 22, 18, 17), lk(23) = C_{14}([**9**, 8, 13, 14, 20, 21, **22**], 18, 17, 4, **3**, 2, 15, 10), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 22, 18, **17**, 16, 20, 21), lk(17) = C_{14}([**16**,15,10,11,12,19,**18**],22,23,3,**4**,5,21,20), lk(9) = C_{14}([**8**, 13, 14, 20, 21, 22, **23**], 3, 2, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 2), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 4, 5, 21, **20**, 14, 1, 2), lk(20) = C_{14}([**21**, 22, 23, 9, 8, 13, **14**], 1, 2, 15, **16**, 17, 4, 5), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 5, 21, **22**, 23, 3, 4), lk(22) = C_{14}([**23**, 9, 8, 13, 14, 20, **21**], 5, 6, 19, **18**, 17, 4, 3). This map is isomorphic to $N_1(6^2, 8)$ by the map (0, 13)(1, 14)(2, 20)(3, 21)(4, 22)(5, 23)(6, 9)(7, 8)(10, 19)(11, 12)(15, 18)(16, 17).

Case 3 : If $a = 20$ then we see that $b \in \{13, 14, 21\}$, *i.e.*, $(a, b) \in \{(20, 13), (20, 14), (20, 21)\}$. If $(a, b) = (20, 13)$ then successively we get $c = 14$, $d = 21$, $e = 22$ and $f = 23$. This implies lk(8) = C_{14}([**20**, 13, 14, 21, 22, 23, **9**], 10, 11, 0, **7**, 6, h, g), where $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17)\}$. If $(g, h) \in \{(17, 16), (17, 18)\}$ then considering lk(8) and lk(6) successively we see that lk(15) or lk(19) can not be completed. For the remaining values of (g, h) we have following :

If $(g, h) = (16, 15)$ then lk(8) = C_{14}([**9**, 23, 22, 21, 14, 13, **20**], 16, 15, 6, **7**, 0, 11, 10). This implies lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **15**, 10, 9, 23), lk(9) = C_{14}([**23**, 22, 21, 14, 13, 20, **8**], 7, 0, 11, **10**, 15, 6, 5) and lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 15, 10, 9, **23**, 22, i, j) for some $i, j \in V(O_2)$. Observe that $(i, j) \in \{(17, 18), (18, 17), (18, 19)\}$. If $(i, j) = (17, 18)$ then considering lk(22) we see 13 21 as an edge and a non-edge both and if $(i, j) = (18, 17)$ or (18, 19) then successively considering lk(5) and lk(4) we see $\deg(13) > 3$ or $\deg(20) > 3$. So $(g, h) \neq (16, 15)$. If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 21, 14, 13, **20**], 16, 17, 6, **7**, 0, 11, 10) and lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, j, i) for some $i, j \in V(O_3)$. In this case $(j, i) \in \{(21, 22), (22, 21), (22, 23), (23, 22)\}$. If $(j, i) = (21, 22)$ then successively considering lk(6) and lk(5) we see that lk(23) can not be completed. If $(j, i) = (22, 21)$ then we see that $0, 1 \in V(O_2)$. A contradiction, as $0, 1 \in V(O_1)$. If $(j, i) = (22, 23)$ then considering lk(6), lk(5) and lk(4) successively we get $\deg(13) > 3$. If $(j, i) = (23, 22)$ then considering lk(18) we see that 15 19 is simultaneously an edge and a non-edge of N. So $(g, h) \neq (16, 17)$. If $(g, h) = (18, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 21, 14, 13, **20**], 18, 17, 6, **7**, 0, 11, 10), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. Now proceeding as in previous case, we see that the map does not exist.

Subcase 3.1 : If $(a, b) = (20, 14)$ then successively we get $c = 13$, $d = 21$, $e = 22$ and $f = 23$. This implies lk(8) $= C_{14}([\mathbf{20}, 14, 13, 21, 22, 23, \mathbf{9}], 10, 11, 0, \mathbf{7}, 6, h, g)$ for some $g, h \in V(O_2)$. In this case $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17), (18, 19)\}$. If $(g, h) \in \{(17, 16), (17, 18)\}$ then considering lk(8) and lk(6) successively we see lk(15) or lk(19) can not be completed. For the remaining values of (g, h) we have following :
Subcase 3.1.1 : If $(g, h) = (16, 15)$ then lk(8) $= C_{14}([\mathbf{20}, 14, 13, 21, 22, 23, \mathbf{9}], 10, 11, 0, \mathbf{7}, 6, 15, 16)$, lk(7) $= C_{14}([\mathbf{0}, 1, 2, 3, 4, 5, \mathbf{6}], 15, 16, 20, \mathbf{8}, 9, 10, 11)$, lk(6) $= C_{14}([\mathbf{5}, 4, 3, 2, 1, 0, \mathbf{7}], 8, 20, 16, \mathbf{15}, 10, 9, 23)$, lk(15) $= C_{14}([\mathbf{16}, 17, 18, 19, 12, 11, \mathbf{10}], 9, 23, 5, \mathbf{6}, 7, 8, 20)$, and lk(5) $= C_{14}([\mathbf{4}, 3, 2, 1, 0, 7, \mathbf{6}], 15, 10, 9, \mathbf{23}, 22, i, j)$ for some $i, j \in V(O_2)$. Observe that $(i, j) \in \{(18, 19), (19, 18)\}$. If $(i, j) = (19, 18)$ then considering lk(5) we see that lk(22) can not be completed. On the other hand when $(i, j) = (18, 19)$ then lk(23) $= C_{14}([\mathbf{9}, 8, 20, 14, 13, 21, \mathbf{22}], 18, 19, 4, \mathbf{5}, 6, 15, 10)$, completing successively we get lk(1) $= C_{14}([\mathbf{0}, 7, 6, 5, 4, 3, \mathbf{2}], 17, 16, 20, \mathbf{14}, 13, 12, 11)$, lk(14) $= C_{14}([\mathbf{20}, 8, 9, 23, 22, 21, \mathbf{13}], 12, 11, 0, \mathbf{1}, 2, 17, 16)$, lk(2) $= C_{14}([\mathbf{3}, 4, 5, 6, 7, 0, \mathbf{1}], 14, 20, 16, \mathbf{17}, 18, 22, 21)$, lk(17) $= C_{14}([\mathbf{16}, 15, 10, 11, 12, 19, \mathbf{18}], 22, 21, 3, \mathbf{2}, 1, 14, 20)$, lk (3) $= C_{14}([\mathbf{4}, 5, 6, 7, 0, 1, \mathbf{2}], 17, 18, 22, \mathbf{21}, 13, 12, 19)$, lk(21) $= C_{14}([\mathbf{13}, 14, 20, 8, 9, 23, \mathbf{22}], 18, 17, 2, \mathbf{3}, 4, 19, 12)$, lk(4) $= C_{14}([\mathbf{5}, 6, 7, 0, 1, 2, \mathbf{3}], 21, 13, 12, \mathbf{19}, 18, 22, 23)$, lk(19) $= C_{14}([\mathbf{12}, 11, 10, 15, 16, 17, \mathbf{18}], 22, 23, 5, \mathbf{4}, 3, 21, 13)$, lk(9) $= C_{14}([\mathbf{8}, 20, 14, 13, 21, 22, \mathbf{23}], 5, 6, 15, \mathbf{10}, 11, 0, 7)$, lk(10) $= C_{14}([\mathbf{15}, 16, 17, 18, 19, 12, \mathbf{11}], 0, 7, 8, \mathbf{9}, 23, 5, 6)$, lk(12) $= C_{14}([\mathbf{11}, 10, 15, 16, 17, 18, \mathbf{19}], 4, 3, 21, \mathbf{13}, 14, 1, 0)$, lk(13) $= C_{14}([\mathbf{14}, 20, 8, 9, 23, 22, \mathbf{21}], 3, 4, 19, \mathbf{12}, 11, 0, 1)$, lk(16) $= C_{14}([\mathbf{15}, 10, 11, 12, 19, 18, \mathbf{17}], 2, 1, 14, \mathbf{20}, 8, 7, 6)$, lk(20) $= C_{14}([\mathbf{8}, 9, 23, 22, 21, 13, \mathbf{14}], 1, 2, 17, \mathbf{16}, 15, 6, 7)$, lk(18) $= C_{14}([\mathbf{17}, 16, 15, 10, 11, 12, \mathbf{19}], 4, 5, 23, \mathbf{22}, 21, 3, 2)$, lk(22) $= C_{14}([\mathbf{23}, 9, 8, 20, 14, 13, \mathbf{21}], 3, 2, 17, \mathbf{18}, 19, 4, 5)$. This is isomorphic to $N_1(6^2, 8)$ by the map (0, 14, 18, 4, 23, 10)(1, 20, 19, 3, 22, 15, 7, 13, 17, 5, 9, 11)(2, 21, 16, 6, 8, 12).
Subcase 3.1.2 : If $(g, h) = (16, 17)$ then lk(8) $= C_{14}([\mathbf{20}, 14, 13, 21, 22, 23, \mathbf{9}], 10, 11, 0, \mathbf{7}, 6, 17, 16)$, lk(7) $= C_{14}([\mathbf{0}, 1, 2, 3, 4, 5, \mathbf{6}], 17, 16, 20, \mathbf{8}, 9, 10, 11)$, lk(6) $= C_{14}([\mathbf{5}, 4, 3, 2, 1, 0, \mathbf{7}], 8, 20, 16, \mathbf{17}, 18, i, j)$ for some $i, j \in V(O_3)$. We see easily that $(i, j) \in \{(21, 22), (22, 21), (22, 23), (23, 22)\}$. If $(i, j) = (21, 22)$ then successively considering lk(6) and lk(5) we see that lk(23) can not be completed. If $(i, j) = (23, 22)$ then considering lk(6) we see that lk(18) can not be completed. If $(i, j) = (22, 23)$ then successively considering lk(6),lk(5) and lk(4) we get deg(14) > 3, a contradiction. If $(i, j) = (22, 21)$ then lk(17) $= C_{14}([\mathbf{16}, 15, 10, 11, 12, 19, \mathbf{18}], 22, 21, 5, \mathbf{6}, 7, 8, 20)$, completing successively we get lk(5) $= C_{14}([\mathbf{4}, 3, 2, 1, 0, 7, \mathbf{6}], 17, 18, 22, \mathbf{21}, 13, 12, 19)$, lk(21) $= C_{14}([\mathbf{13}, 14, 20, 8, 9, 23, \mathbf{22}], 18, 17, 6, \mathbf{5}, 4, 19, 12)$, lk(4) $= C_{14}([\mathbf{5}, 6, 7, 0, 1, 2, \mathbf{3}], 23, 22, 18, \mathbf{19}, 12, 13, 21)$, lk(19) $= C_{14}([\mathbf{12}, 11, 10, 15, 16, 17, \mathbf{18}], 22, 23, 3, \mathbf{4}, 5, 21, 13)$, lk(1) $= C_{14}([\mathbf{0}, 7, 6, 5, 4, 3, \mathbf{2}], 15, 16, 20, \mathbf{14}, 13, 12, 11)$, lk (14) $= C_{14}([\mathbf{20}, 8, 9, 23, 22, 21, \mathbf{13}], 12, 11, 0, \mathbf{1}, 2, 15, 16)$, lk(2) $= C_{14}([\mathbf{3}, 4, 5, 6, 7, 0, \mathbf{1}], 14, 20, 16, \mathbf{15}, 10, 9, 23)$, lk(15) $= C_{14}([\mathbf{16}, 17, 18, 19, 12, 11, \mathbf{10}], 9, 23, 3, \mathbf{2}, 1, 14, 20)$, lk(3) $= C_{14}([\mathbf{4}, 5, 6, 7, 0, 1, \mathbf{2}], 15, 10, 9, \mathbf{23}, 22, 18, 19)$, lk(23) $= C_{14}([\mathbf{9}, 8, 20, 14, 13, 21, \mathbf{22}], 18, 19, 4, \mathbf{3}, 2, 15, 10)$, lk(9) $= C_{14}([\mathbf{8}, 20, 14, 13, 21, 22, \mathbf{23}], 3, 2, 15, \mathbf{10}, 11, 0, 7)$, lk(10) $= C_{14}([\mathbf{15}, 16, 17, 18, 19, 12, \mathbf{11}], 0, 7, 8, \mathbf{9}, 23, 3, 2)$,

lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 4, 5, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 20, 8, 9, 23, 22, **21**], 5, 4, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 6, 7, 8, **20**, 14, 1, 2), lk(20) = C_{14}([**8**, 9, 23, 22, 21, 13, **14**], 1, 2, 15, **16**, 17, 6, 7), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 4, 3, 23, **22**, 21, 5, 6), lk(22) = C_{14}([**23**, 9, 8, 20, 14, 13, **21**], 5, 6, 17, **18**, 19, 4, 3). This is $N_2(6^2, 8)$ as given in Section 2.
Subcase 3.1.3: If $(g, h) = (18, 17)$ then lk(8) = C_{14}([**20**, 14, 13, 21, 22, 23, **9**], 10, 11, 0, **7**, 6, 17, 18), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 17, 18, 20, **8**, 9, 10, 11). This implies lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$, observe that $(i, j) \in \{(21, 22), (22, 21), (22, 23)\}$. If $(i, j) = (21, 22)$ then considering lk(6) we see that lk(5) and lk(23) can not be completed. If $(i, j) = (22, 21)$ then successively considering lk(6), lk(5), lk(21), lk(13), lk(12), lk(19) and lk(4) we get deg(14) > 3. A contradiction. If $(i, j) = (22, 23)$ then lk(6) = C_{14}([**7**, 0, 1, 2, 3, 4, **5**], 23, 22, 16, **17**, 18, 20, 8), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 8, 7, **6**, 5, 23, 22), completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 19, 18, 20, **14**, 13, 12, 11), lk(14) = C_{14}([**20**, 8, 9, 23, 22, 21, **13**], 12, 11, 0, **1**, 2, 19, 18), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 20, 18, **19**, 12, 13, 21), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 20, 14, 1, **2**, 3, 21, 13), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 19, 12, 13, **21**, 22, 16, 15), lk(21) = C_{14}([**13**, 14, 20, 8, 9, 23, **22**], 16, 15, 4, **3**, 2, 19, 12), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 21, 22, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 5, **4**, 3, 21, 22), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 16, 22, **23**, 9, 10, 15), lk(23) = C_{14}([**9**, 8, 20, 14, 13, 21, **22**], 16, 17, 6, **5**, 4, 15, 10), lk(9) = C_{14}([**8**, 20, 14, 13, 21, 22, **23**], 5, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 5, 4), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 2, 3, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 20, 8, 9, 23, 22, **21**], 3, 2, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 6, 5, 23, **22**, 21, 3, 4), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 2, 1, 14, **20**, 8, 7, 6), lk(20) = C_{14}([**8**, 9, 23, 22, 21, 13, **14**], 1, 2, 19, **18**, 17, 6, 7), lk(22) = C_{14}([**23**, 9, 8, 20, 14, 13, **21**], 3, 4, 15, **16**, 17, 6, 5). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 21, 10, 6, 14, 16)(1, 22, 11, 5, 13, 15, 7, 20, 17)(2, 23, 12, 4, 8, 18)(3, 9, 19).
Subcase 3.1.4: If $(g, h) = (18, 19)$ then lk(8) = C_{14}([**20**, 14, 13, 21, 22, 23, **9**], 10, 11, 0, **7**, 6, 19, 18), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**],19, 18, 20, **8**, 9, 10, 11), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **19**, 12, 13, 21), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 20, 8, 7, **6**, 5, 21, 13) and lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 12, 13, **21**, 22, i, j) for some $i, j \in V(O_2)$. It is easy to see that $(i, j) \in \{(15, 16), (16, 15), (16, 17), (17, 16)\}$. If $(i, j) = (17, 16)$ then considering lk(17) we see that 14 23 is simultaneously an edge and a non-edge of N. If $(i, j) = (15, 16)$ then considering lk(5) we see that lk(4) and lk(17) can not be completed. If $(i, j) = (16, 17)$ then successively considering lk(5) and lk(4) we get deg(14) > 3. A contradiction. If $(i, j) = (16, 15)$ then lk(21) = C_{14}([**13**, 14, 20, 8, 9, 23, **22**], 16, 15, 4, **5**, 6, 19, 12), completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 17, 18, 20, **14**, 13, 12, 11), lk(14) = C_{14}([**20**, 8, 9, 23, 22, 21, **13**], 12, 11, 0, **1**, 2, 17, 18), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 20, 18, **17**, 16, 22, 23), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 14, 1, **2**, 3, 23, 22), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 17, 16, 22, **23**, 9, 10, 15), lk(23) = C_{14}([**9**, 8, 20, 14, 13, 21, **22**], 16, 17, 2, **3**, 4, 15, 10), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 9, 10, **15**,

16, 22, 21), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**],9, 23, 3, **4**, 5, 21, 22), lk(9) = C_{14}([**8**, 20, 14, 13, 21, 22, **23**], 3, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14} ([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 4), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 6, 5, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 20, 8, 9, 23, 22, **21**], 5, 6, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 2, 3, 23, **22**, 21, 5, 4), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 7, 8, **20**, 14, 1, 2), lk(20) = C_{14}([**8**, 9, 23, 22, 21, 13, **14**], 1, 2, 17, **18**, 19, 6, 7), lk(22) = C_{14}([**23**, 9, 8, 20, 14, 13, **21**], 5, 4, 15, **16**, 17, 2, 3). This is isomorphic to $N_2(6^2, 8)$ by the map (0, 1, 2, 3, 4, 5, 6, 7)(8, 11, 14, 15, 21, 17, 23, 19)(9, 12, 20, 10, 13, 16, 22, 18).

Subcase 3.2 : If $(a, b) = (20, 21)$ then we see that $c \in \{13, 14, 22\}$.

If $c = 14$ then completing successively we get $d = 13$, $e = 22$, $f = 23$ and $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17), (18, 19)\}$. If $(g, h) = (16, 15)$ then considering lk(8), lk(9) we see that lk(5) and lk(22) can not be completed. If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 13, 14, 21, **20**], 16, 17, 6, **7**, 0, 11, 10) and lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, i, j) for some $i, j \in V(O_3)$. Observe that $(i, j) \in \{(22, 23), (23, 22)\}$. If $(i, j) = (22, 23)$ then successively considering lk(5), lk(22), lk(13), lk(4), lk(12), lk(19), lk(18) and lk(23) we see that lk(2) and lk(3) can not be completed. If $(i, j) = (23, 22)$ then successively considering lk(5), lk(22), lk(13), lk(4), lk(12) and lk(18) we see that lk(2) and lk(3) can not be completed. If $(g, h) = (17, 16)$ then lk(8) = C_{14}([**9**, 23, 22, 13, 14, 21, **20**], 17, 16, 6, **7**, 0, 11, 10), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 17, **16**, 15, j, i) and lk(15) = C_{14}([**10**, 11, 12, 19, 18, 17, **16**], 6, 5, i, $\boldsymbol{j}$, k, 23, 9) for some $i, j, k \in V(O_3)$. Now, it is easy to see that i, j, k have no values in V so that lk(15) can be completed. In case $(g, h) = (17, 18)$ then considering lk(8) we see lk(19) can not be completed. If $(g, h) = (18, 19)$ then considering lk(8) and lk(6) we see that lk(23) can not be completed. If $(g, h) = (18, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 13, 14, 21, **20**], 18, 17, 6, **7**, 0, 11, 10), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. In this case $(i, j) \in \{(22, 23), (23, 22)\}$. If $(i, j) = (22, 23)$ then successively considering lk(6), lk(5), lk(9), lk(10), lk(23) and lk(4) we get deg(13) > 3. If $(i, j) = (23, 22)$ then successively considering lk(6), lk(5), lk(4), lk(13), lk(12), lk(19), lk(18) and lk(20) we see that length of lk(2) is less than 14. So $c \neq 14$.

Subcase 3.2.1 : If $c = 13$ then $d = 14$, $e = 22$, $f = 23$ and $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17), (18, 19)\}$. If $(g, h) \in \{(17, 16), (17, 18)\}$ then considering lk(8) and lk(6) we see lk(15) or lk(19) can not be completed. For the remaining values of (g, h) we have following subcases.

Subcase 3.2.1.1 : If $(g, h) = (18, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 14, 13, 21, **20**], 18, 17, 6, **7**, 0, 11, 10) and lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. Observe that $i \in \{22, 23\}$. If $i = 22$ then $j = 23$, now successively considering lk(5),lk(10), lk(9) and lk(4) we see deg(14) > 3. A contradiction. If $i = 23$ then $j = 22$ this implies length of lk(23) is less than 14. A contradiction. So $(g, h) \neq (18, 17)$.

Subcase 3.2.1.2 : If $(g, h) = (16, 15)$ then lk(8) = C_{14}([**9**, 23, 22, 21, 13, 14, **20**], 16, 15, 6, **7**, 0, 11, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 15, 16, 20, **8**, 9, 10, 11), lk(6) = C_{14}([**7**, 0, 1, 2, 3, 4, **5**],23, 9, 10, **15**, 16, 20, 8), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 5, **6**, 7, 8, 20), lk(5) = C_{14} ([**4**, 3, 2, 1, 0, 7, **6**],15, 10, 9, **23**, 22, i, j), lk(22) = C_{14}([**23**, 9, 8, 20, 21, 13, **14**], 21, 20, k, $\boldsymbol{i}$, j, 4, 5) for some $i, j, k \in V(O_2)$. In this case $i \in \{17, 18, 19\}$. If $i = 17$ then $j = 18$, now considering lk(17) we see that 14 21 is both an edge and a non-edge. If $i = 19$ then $j = 18$ and $k = 12$, now considering lk(22) we see that 13 14 is both an edge and a non-edge of N. If $i = 18$ then $j = 17$ and $k = 19$. This implies lk(23) = C_{14}([**9**, 8, 20, 21, 13, 14, **22**], 18, 17, 4, **5**, 6, 15, 10), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 2, 1, 14, **22**, 23, 5, 4). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 19, 18, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 21, 20, 8, 9, 23, **22**],18, 19, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 18, **19**, 12, 13, 21), lk(19) = C_{14} ([**12**, 11, 10, 15, 16, 17, **18**], 22, 14, 1, **2**, 3, 21, 13), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 19, 12, 13, **21**, 20, 16, 17), lk(21) = C_{14}([**13**, 14, 22, 23, 9, 8, **20**], 16, 17, 4, **3**, 2, 19, 12), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 21, 20, 16, **17**, 18, 22, 23), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 22, 23, 5, **4**, 3, 21, 20), lk(9) = C_{14}([**8**, 20, 21, 13, 14, 22, **23**], 5, 6, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 5, 6), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 2, 3, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 23, 9, 8, 20, **21**], 3, 2, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 4, 3, 21, **20**, 8, 7, 6), lk (20) = C_{14}([**8**, 9, 23, 22, 14, 13, **21**], 3, 4, 17, **16**, 15, 6, 7). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 10, 23, 2, 12, 8, 4, 18, 14)(1, 11, 9, 3, 19, 13, 7, 15, 22)(5, 17, 20)(6, 16, 21).

Subcase 3.2.1.3 : If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 23, 22, 14, 13, 21, **20**], 16, 17, 6, **7**, 0, 11, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 17, 16, 20, **8**, 9, 10, 11) and lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, i, j) for some $i, j \in V(O_3)$. Observe that $i = 22$, this implies $j = 23$. Then lk(6) = C_{14}([**7**, 0, 1, 2, 3, 4, **5**], 23, 22, 18, **17**, 16, 20, 8), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 22, 23, 5, **6**, 7, 8, 20). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 19, 18, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 21, 20, 8, 9, 23, **22**], 18, 19, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 18, **19**, 12, 13, 21), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 22, 14, 1, **2**, 3, 21, 13), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 19, 12, 13, **21**, 20, 16, 15), lk(21) = C_{14}([**13**, 14, 22, 23, 9, 8, **20**], 16, 15, 4, **3**, 2, 19, 12), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 21, 20, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 5, **4**, 3, 21, 20), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 18, 22, **23**, 9, 10, 15), lk(23) = C_{14}([**9**, 8, 20, 21, 13, 14, **22**], 18, 17, 6, **5**, 4, 15, 10), lk(9) = C_{14}([**8**, 20, 21, 13, 14, 22, **23**], 5, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 5, 4), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 2, 3, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 23, 9, 8, 20, **21**], 3, 2, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 6, 7, 8, **20**, 21, 3, 4), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 2, 1, 14, **22**, 23, 5, 6), lk(20) = C_{14}([**8**, 9, 23, 22, 14, 13, **21**], 3, 4, 15, **16**, 17, 6, 7), lk(22) = C_{14}([**23**, 9, 8, 20, 21, 13, **14**],1, 2, 19, **18**, 17, 6, 5). This is isomorphic to $N_2(6^2, 8)$ by the map (0, 3, 6, 1, 4, 7, 2, 5)(8, 15)(9, 10)(11, 23)(12, 22)(13, 18)(14, 19, 21, 17)(16, 20).

Subcase 3.2.1.4 : If $(g, h) = (18, 19)$ then completing lk(6), lk(5), lk(21), lk(20) we get lk(4) = C_{14}([**3**, 2, 1, 0, 7, 6, **5**], 21, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. Observe that $i = 22$, this implies $j = 23$. Then lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 21, 5, **4**, 3, 23, 22). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 15, 16, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 21, 20, 8, 9, 23, **22**], 16, 15, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **2**, 1, 14, 22), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 15, 10, 9, **23**, 22, 16, 17), lk(23) = C_{14}([**9**, 8, 20, 21, 13, 14, **22**], 16, 17, 4, **3**, 2, 15, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 19, 18, 20, **8**, 9, 10, 11), lk(9) = C_{14}([**8**, 20, 21, 13, 14, 22, **23**], 3, 2, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 2), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 6, 5, 21, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 23, 9, 8, 20, **21**], 5, 6, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 4, 3, 23, **22**, 14, 1, 2), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 7, 8, **20**, 21, 5, 4), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 20, 8, 7, **6**, 5, 21, 13), lk (22) = C_{14}([**23**, 9, 8, 20, 21, 13, **14**], 1, 2, 15, **16**, 17, 4, 3). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 20)(1, 21, 7, 14, 5, 8)(2, 22, 4, 9)(3, 23)(6, 13)(10, 17)(11, 18)(12, 19)(15, 16).

Subcase 3.2.2 : If $c = 22$ then we have $d \in \{13, 14, 23\}$.

If $d = 13$ then successively we get $e = 14$, $f = 23$ and $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17)\}$. If $(g, h) \in \{(17, 16), (17, 18)\}$ then considering lk(8) we see that lk(6) can not be completed. If $(g, h) = (16, 15)$ then successively considering lk(8), lk(6) and lk(9) we see that lk(14) and lk(23) can not be completed. If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 23, 14, 13, 22, 21, **20**], 16, 17, 6, **7**, 0, 11, 10), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, i, j) for some $i, j \in V(O_3)$. Observe that $(i, j) \in \{(21, 22), (22, 21)\}$. If $(i, j) = (21, 22)$ then successively considering lk(6), lk(5) and lk(4) we see that $\deg(20) > 3$. A contradiction. If $(i, j) = (22, 21)$ then successively we get lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, 22, 21), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 18, 22, **21**, 20, 16, 15), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 9, 10, **15**, 16, 20, 21) and lk(9) = C_{14}([**8**, 20, 21, 22, 13, 14, **23**], 3, 4, 15, **10**, 11, 0, 7). This implies $C_9(0, 1, 14, 23, 3, 4, 5, 6, 7) \subseteq$ lk(2), a contradiction. If $(g, h) = (18, 17)$ then lk(8) = C_{14}([**9**, 23, 14, 13, 22, 21, **20**], 18, 17, 6, **7**, 0, 11, 10), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. In this case, $(i, j) \in \{(21, 22), (22, 21)\}$. Now proceeding further we get a contradiction for each value of (i, j). So $d \neq 13$.

If $d = 14$ then $e = 13$, $f = 23$ and $(g, h) \in \{(16, 15), (16, 17), (17, 16), (17, 18), (18, 17), (18, 19)\}$. If $(g, h) \in \{(17, 16), (17, 18)\}$ then considering lk(8) we see lk(15) or lk(19) can not be completed. For the remaining values of (g, h) we have following subcases.

Subcase A: If $(g, h) = (16, 15)$ then successively we get lk(8) = C_{14}([**9**, 23, 13, 14, 22, 21, **20**], 16, 15, 6, **7**, 0, 11, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 15, 16, 20, **8**, 9, 10, 11), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**10**, 11, 12, 19, 18, 17, **16**], 20, 8, 7, **6**, 5, 23, 9), lk(23) = C_{14}([**9**, 8, 20, 21, 22, 14, **13**], 12, 19, 4, **5**, 6, 15, 10), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 15, 10, 9, **23**, 13, 12, 19), lk(4) = C_{14}([**3**, 2, 1, 0, 7, 6, **5**], 23, 13, 12, **19**, 18, i, j) for some $i, j \in V(O_3)$. Observe that $i \in \{21, 22\}$. If $i = 21$ then $j = 22$. This implies $C_9(0, 1, 14, 22, 3, 4, 5, 6, 7) \subseteq$ lk(2). A contradiction. So $i = 22$ then $j = 21$. This implies lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 21, 22, 18, **19**, 12, 13, 23), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 22, 21, 3, **4**, 5, 23, 13). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 17, 18, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 23, 9, 8, 20, 21, **22**], 18, 17, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 18, **17**, 16, 20, 21), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 22, 14, 1, **2**, 3, 21, 20), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 17, 16, 20, **21**, 22, 18, 19), lk(21) = C_{14}([**22**, 14, 13, 23, 9, 8, **20**], 16, 17, 2, **3**, 4, 19, 18), lk(9) = C_{14}([**8**, 20, 21, 22, 14, 13, **23**], 5, 6, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 5, 6), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 4, 5, 23, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 21, 20, 8, 9, **23**], 5, 4, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 2, 3, 21, **20**, 8, 7, 6), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 4, 3, 21, **22**, 14, 1, 2), lk(20) = C_{14}([**8**, 9, 23, 13, 14, 22, **21**], 3, 2, 17, **16**, 15, 6, 7), lk(22) = C_{14}([**14**, 13, 23, 9, 8, 20, **21**], 3, 4, 19, **18**, 17, 2, 1). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 1, 2, 3, 4, 5, 6, 7)(8, 11, 14, 17, 23, 19, 21, 15)(9, 12, 20, 10, 13, 18, 22, 16).

Subcase B: If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 23, 13, 14, 22, 21, **20**], 16, 17, 6, **7**, 0, 11, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 17, 16, 20, **8**, 9, 10, 11), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 16, **17**, 18, i, j) for some $i, j \in V(O_3)$. In this case $i \in \{21, 22\}$. If $i = 21$, $j = 22$. Now considering lk(21) we see 15 19 as an edge and a non-edge both. If $i = 22$ then $j = 21$. This implies lk(6) = C_{14} ([**7**, 0, 1, 2, 3, 4, **5**], 21, 22, 18, **17**, 16, 20, 8), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 22, 21, 5, **6**, 7, 8, 20). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 19, 18, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 23, 9, 8, 20, 21, **22**], 18, 19, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 18, **19**, 12, 13, 23), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 22, 14, 1, **2**, 3, 23, 13), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 19, 12, 13, **23**, 9, 10, 15), lk (23) = C_{14}([**9**, 8, 20, 21, 22, 14, **13**], 12, 19, 2, **3**, 4, 15, 10), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 9, 10, **15**, 16, 20, 21), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **4**, 5, 21, 20), lk(21) = C_{14} ([**22**, 14, 13, 23, 9, 8, **20**], 16, 15, 4, **5**, 6, 17, 18), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 18, 22, **21**, 20, 16, 15), lk(9) = C_{14}([**8**, 20, 21, 22, 14, 13, **23**], 3, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 4), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 2, 3, 23, **13**, 14, 1, 0), lk (13) = C_{14}([**14**, 22, 21, 20, 8, 9, **23**], 3, 2, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 6, 7, 8, **20**, 21, 5, 4), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 2, 1, 14, **22**, 21, 5, 6), lk(20) = C_{14}([**8**, 9, 23, 13, 14, 22, **21**], 5, 4, 15, **16**, 17, 6, 7), lk(22) = C_{14}([**14**, 13, 23, 9, 8, 20, **21**], 5, 6, 17, **18**, 19, 2, 1). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 2, 4, 6)(1, 3, 5, 7)(8, 14, 23, 21)(9, 20, 13, 22)(10, 18)(11, 17)(12, 16)(15, 19).

Subcase C: If $(g, h) = (18, 17)$ then successively we get lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, i, j) for some $i, j \in V(O_3)$. In this case $(i, j) \in \{(21, 22), (22, 21)\}$. If $(i, j) = (21, 22)$ then considering lk(21) we see that 15 19 is both an edge and a non-edge. So $(i, j) = (22, 21)$ then lk(6) = C_{14}([**7**, 0, 1, 2, 3, 4, **5**], 21, 22, 16, **17**, 18, 20, 8), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 8, 7, **6**, 5, 21, 22). Now completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 15, 16, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 23, 9, 8, 20, 21, **22**], 16, 15, 2, **1**, 0, 11, 12), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 3, **2**, 1, 14, 22), lk(23) = C_{14}([**9**, 8, 20, 21, 22, 14, **13**], 12, 19, 4, **3**, 2, 15, 10), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 15, 10, 9, **23**, 13, 12, 19), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**],20, 21, 5, **4**, 3, 23, 13), lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 23, 13, 12, **19**, 18, 20, 21), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 16, 22, **21**, 20, 18, 19), lk(21) = C_{14}([**22**, 14, 13, 23, 9, 8, **20**], 18, 19, 4, **5**, 6, 17, 16), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 17, 18, 20, **8**, 9, 10, 11), lk(9) = C_{14}([**8**, 20, 21, 22, 14, 13, **23**],3, 2, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 3, 2), lk(12) = C_{14} ([**11**, 10, 15, 16, 17, 18, **19**], 4, 3, 23, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 21, 20, 8, 9, **23**], 3, 4, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 6, 5, 21, **22**, 14, 1, 2), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 4, 5, 21, **20**, 8, 7, 6), lk(20) = C_{14}([**8**, 9, 23, 13, 14, 22, **21**], 5, 4, 19, **18**, 17, 6, 7), lk(22) = C_{14}([**14**, 13, 23, 9, 8, 20, **21**], 5, 6, 17, **16**, 15, 2, 1). This is isomorphic to $N_2(6^2, 8)$ by the map (0, 7, 6, 5, 4, 3, 2, 1)(8, 17, 21, 19, 23, 15, 14, 11)(9, 16, 13, 10, 20, 18, 22, 12).

Subcase D: If $(g, h) = (18, 19)$ then successively we get lk(8) = C_{14}([**9**, 23, 13, 14, 22, 21, **20**], 18, 19, 6, **7**, 0, 11, 10), lk(7) = C_{14}([**0**, 1, 2, 3, 4, 5, **6**], 19, 18, 20, **8**, 9, 10, 11), lk(19) = C_{14}([**12**, 11, 10, 15, 16, 17, **18**], 20, 8, 7, **6**, 5, 23, 13), lk(6) = C_{14}([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **19**, 12, 13, 23), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 19, 12, 13, **23**, 9, 10, 15), lk(23) = C_{14}([**9**, 8, 20, 21, 22, 14, **13**], 12, 19, 6, **5**, 4, 15, 10), lk(4) = C_{14}([**3**, 2, 1, 0, 7, 6, **5**], 23, 9, 10, **15**, 16, i, j) for some $i, j \in V(O_3)$. In this case we see, $(i, j) = (22, 21)$. Then lk(4) = C_{14}([**5**, 6, 7, 0, 1, 2, **3**], 21, 22, 16, **15**, 10, 9, 23), lk(15) = C_{14}([**16**, 17, 18, 19, 12, 11, **10**], 9, 23, 5, **4**, 3, 21, 22), completing successively we get lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 17, 16, 22, **14**, 13, 12, 11), lk(14) = C_{14}([**13**, 23, 9, 8, 20, 21, **22**], 16, 17, 2, **1**, 0, 11, 12), lk(17) = C_{14}([**16**, 15, 10, 11, 12, 19, **18**], 20, 21, 3, **2**, 1, 14, 22), lk(2) = C_{14}([**3**, 4, 5, 6, 7, 0, **1**], 14, 22, 16, **17**, 18, 20, 21), lk(21) = C_{14}([**22**, 14, 13, 23, 9, 8, **20**], 18, 17, 2, **3**, 4, 15, 16), lk(3) = C_{14}([**4**, 5, 6, 7, 0, 1, **2**], 17, 18, 20, **21**, 22, 16, 15), lk(9) = C_{14} ([**8**, 20, 21, 22, 14, 13, **23**], 5, 4, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**15**, 16, 17, 18, 19, 12, **11**], 0, 7, 8, **9**, 23, 5, 4), lk(12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 6, 5, 23, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 22, 21, 20, 8, 9, **23**], 5, 6, 19, **12**, 11, 0, 1), lk(16) = C_{14}([**15**, 10, 11, 12, 19, 18, **17**], 2, 1, 14, **22**, 21, 3, 4), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 6, 7, 8, **20**, 21, 3, 2), lk(20) = C_{14}([**8**, 9, 23, 13, 14, 22, **21**], 3, 2, 17, **18**, 19, 6, 7), lk(22) = C_{14}([**14**, 13, 23, 9, 8, 20, **21**], 3, 4, 15, **16**, 17, 2, 1). This is isomorphic to $N_2(6^2, 8)$ by the map (0, 2, 4, 6)(1, 3, 5, 7)(8, 14, 23)(9, 20, 13)(10, 16, 18, 12)(11, 15, 17, 19).

Subcase 3.2.2.2: If $d = 23$ then $(e, f) = (13, 14)$. This implies lk(14) = C_{14}([**9**, 8, 20, 21, 22, 23, **13**], 12, 11, 0, **1**, 2, 15, 10), lk(1) = C_{14}([**0**, 7, 6, 5, 4, 3, **2**], 15, 10, 9, **14**, 13, 12, 11), lk(9) = C_{14}([**8**, 20, 21, 22, 23, 13, **14**], 1, 2, 15, **10**, 11, 0, 7), lk(10) = C_{14}([**11**, 12, 19, 18, 17, 16, **15**], 2, 1, 14, **9**, 8, 7, 0) and lk(8) = C_{14}([**20**, 21, 22, 23, 13, 14, **9**], 10, 11, 0, **7**, 6, h, g), where $(g, h) \in \{(16,\ 17),\ (17,\ 16),\ (17,\ 18),\ (18,\ 17),\ (18,\ 19)\}$. If $(g, h) = (17, 16)$ or $(17, 18)$ then considering lk(8) we see that lk(15) or lk(19) can not be completed. Also, $(16, 17) \cong (18, 19)$ by the map (0, 9)(1, 14)(2, 13)(3, 23)(4, 22)(5, 21)(6, 20)(7, 8)(10, 11)(12, 15)(16, 19)(17, 18). So we search the map for $(g, h) \in \{(16,\ 17),\ (18,\ 17)\}$.

Subcase A: If $(g, h) = (16, 17)$ then lk(8) = C_{14}([**9**, 14, 13, 23, 22, 21, **20**], 16, 17, 6, **7**, 0, 11, 10), lk(7) = C_{14} ([**0**, 1, 2, 3, 4, 5, **6**], 17, 16, 20, **8**, 9, 10, 11) and lk(6) = C_{14}([**5**, 4, 3,

2, 1, 0, **7**], 8, 20, 16, **17**, 18, i, j) for some $i, j \in V(O_3)$. It is easy to see that $i = 22$ or 23. If $i = 23$ then $j = 22$. Now considering lk(5) we see 34 as an edge and a non-edge both. So $i = 22$ then $j = 23$. This implies lk(6) $= C_{14}$([**7**, 0, 1, 2, 3, 4, **5**], 23, 22, 18, **17**, 16, 20, 8), lk(17) $= C_{14}$([**16**, 15, 10, 11, 12, 19, **18**], 22, 23, 5, **6**, 7, 8, 20), completing successively we get lk(2) $= C_{14}$([**3**, 4, 5, 6, 7, 0, **1**], 14, 9, 10, **15**, 16, 20, 21), lk(15) $= C_{14}$([**16**, 17, 18, 19, 12, 11, **10**], 9, 14, 1, **2**, 3, 21, 20), lk(21) $= C_{14}$([**22**, 23, 13, 14, 9, 8, **20**], 16, 15, 2, **3**, 4, 19, 18), lk(3) $= C_{14}$([**4**, 5, 6, 7, 0, 1, **2**], 15, 16, 20, **21**, 22, 18, 19), lk(4) $= C_{14}$([**5**, 6, 7, 0, 1, 2, **3**], 21, 22, 18, **19**, 12, 13, 23), lk(23) $= C_{14}$([**22**, 21, 20, 8, 9, 14, **13**], 12, 19, 4, **5**, 6, 17, 18), lk(5) $= C_{14}$([**4**, 3, 2, 1, 0, 7, **6**], 17, 18, 22, **23**, 13, 12, 19), lk(12) $= C_{14}$([**11**, 10, 15, 16, 17, 18, **19**], 4, 5, 23, **13**, 14, 1, 0), lk(13) $= C_{14}$([**14**, 9, 8, 20, 21, 22, **23**], 5, 4, 19, **12**, 11, 0, 1), lk(20) $= C_{14}$([**8**, 9, 14, 13, 23, 22, **21**], 3, 2, 15, **16**, 17, 6, 7), lk(16) $= C_{14}$([**15**, 10, 11, 12, 19, 18, **17**], 6, 7, 8, **20**, 21, 3, 2), lk(22) $= C_{14}$([**23**, 13, 14, 9, 8, 20, **21**], 3, 4, 19, **18**, 17, 6, 5), lk(18) $= C_{14}$([**17**, 16, 15, 10, 11, 12, **19**], 4, 3, 21, **22**, 23, 5, 6), lk(19) $= C_{14}$([**12**, 11, 10, 15, 16, 17, **18**], 22, 21, 3, **4**, 5, 23, 13). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 4)(1, 3)(5, 7)(8, 21, 14, 23)(9, 22, 13)(10, 16, 18, 12)(15, 17, 19, 11).
Subcase B: If $(g, h) = (18, 17)$ then lk(8) $= C_{14}$([**9**, 14, 13, 23, 22, 21, **20**], 17, 18, 6, **7**, 0, 11, 10), lk(7) $= C_{14}$([**0**, 1, 2, 3, 4, 5, **6**], 17, 18, 20, **8**, 9, 10, 11). This implies lk(6) $= C_{14}$([**5**, 4, 3, 2, 1, 0, **7**], 8, 20, 18, **17**, 16, j, i), lk(17) $= C_{14}$([**16**, 15, 10, 11, 12, 19, **18**], 20, 8, 7, **6**, 5, 21, 22) and lk(16) $= C_{14}$([**15**, 10, 11, 12, 19, 18, **17**], 6, 5, i, $\boldsymbol{j}$, k, 3, 2) for some $i, j, k \in V(O_3)$. Then we see, $j \in \{21, 22, 23\}$. If $j = 21$ then either $i = 20$ or $k = 20$. But in both cases, deg(20) > 3. A contradiction. If $j = 23$ then $i = 13$ or $k = 13$, again in both cases, we see deg(13) > 3. If $j = 22$ then $i \in \{21, 23\}$.

If $i = 21$ then $k = 23$. This implies lk(16) $= C_{14}$([**15**, 10, 11, 12, 19, 18, **17**], 6, 5, 21, **22**, 23, 3, 2), lk(22) $= C_{14}$([**23**, 13, 14, 9, 8, 20, **21**], 5, 6, 17, **16**, 15, 2, 3), completing successively we get lk(2) $= C_{14}$([**3**, 4, 5, 6, 7, 0, **1**], 14, 9, 10, **15**, 16, 22, 23), lk(15) $= C_{14}$([**16**, 17, 18, 19, 12, 11, **10**], 9, 14, 1, **2**, 3, 23, 22), lk(23) $= C_{14}$([**22**, 21, 20, 8, 9, 14, **13**], 12, 19, 4, **3**, 2, 15, 16), lk(3) $= C_{14}$([**4**, 5, 6, 7, 0, 1, **2**], 15, 16, 22, **23**, 13, 12, 19), lk(19) $= C_{14}$([**12**, 11, 10, 15, 16, 17, **18**], 20, 21, 5, **4**, 3, 23, 13), lk(4) $= C_{14}$([**5**, 6, 7, 0, 1, 2, **3**], 23, 13, 12, **19**, 18, 20, 21), lk(21) $= C_{14}$([**22**, 23, 13, 14, 9, 8, **20**], 18, 19, 4, **5**, 6, 17, 16), lk(5) $= C_{14}$([**4**, 3, 2, 1, 0, 7, **6**], 17, 16, 22, **21**, 20, 18, 19), lk (12) $= C_{14}$([**11**, 10, 15, 16, 17, 18, **19**], 4, 3, 23, **13**, 14, 1, 0), lk(13) $= C_{14}$([**14**, 9, 8, 20, 21, 22, **23**], 3, 4, 19, **12**, 11, 0, 1), lk(18) $= C_{14}$([**17**, 16, 15, 10, 11, 12, **19**], 4, 5, 21, **20**, 8, 7, 6), lk(20) $= C_{14}$ ([**8**, 9, 14, 13, 23, 22, **21**], 5, 4, 19, **18**, 17, 6, 7). This is isomorphic to $N_1(6^2, 8)$ by the map (0, 13, 6, 20, 2, 9)(1, 8)(3, 23, 5, 21)(4, 22)(7, 14)(10, 11, 12, 19, 18, 17, 16, 15).

If $i = 23$ then $k = 21$. This implies lk(16) $= C_{14}$([**15**, 10, 11, 12, 19, 18, **17**], 6, 5, 23, **22**, 21, 3, 2), lk(22) $= C_{14}$([**23**, 13, 14, 9, 8, 20, **21**], 3, 2, 15, **16**, 17, 6, 5), completing successively we get lk(2) $= C_{14}$([**3**, 4, 5, 6, 7, 0, **1**], 14, 9, 10, **15**, 16, 22, 21), lk(15) $= C_{14}$([**16**, 17, 18, 19, 12, 11, **10**], 9, 14, 1, **2**, 3, 21, 22), lk(21) $= C_{14}$([**22**, 23, 13, 14, 9, 8, **20**], 18, 19, 4, **3**, 2, 15, 16), lk(3) $= C_{14}$([**4**, 5, 6, 7, 0, 1, **2**], 15, 16, 22, **21**, 20, 18, 19), lk(19) $= C_{14}$([**12**, 11, 10, 15, 16, 17, **18**], 20, 21, 3, **4**, 5, 23, 13), lk(4) $= C_{14}$([**5**, 6, 7, 0, 1, 2, **3**], 21, 20, 18, **19**, 12, 13, 23), lk(23) $= C_{14}$([**22**, 21, 20, 8, 9, 14, **13**], 12, 19, 4, **5**, 6, 17, 16), lk(5) = C_{14}([**4**, 3, 2, 1, 0, 7, **6**], 17, 16, 22, **23**, 13, 12, 19), lk (12) = C_{14}([**11**, 10, 15, 16, 17, 18, **19**], 4, 5, 23, **13**, 14, 1, 0), lk(13) = C_{14}([**14**, 9, 8, 20, 21, 22, **23**], 5, 4, 19, **12**, 11, 0, 1), lk(18) = C_{14}([**17**, 16, 15, 10, 11, 12, **19**], 4, 3, 21, **20**, 8, 7, 6), lk(20) = C_{14} ([**8**, 9, 14, 13, 23, 22, **21**], 3, 4, 19, **18**, 17, 6, 7). This is isomorphic to $N_2(6^2, 8)$ by the map (0, 5)(1, 4)(2, 3)(6, 7)(8, 17)(9, 18, 20, 16)(10, 22)(11, 21, 15, 23)(12, 13)(14, 19). Thus the Lemma 1.3 is proved. □

Table below shows a list of semi-equivelar maps on the surface of Euler characteristic −1 obtained in this article and in [23].

Table : Semi-equivelar maps on the surface of Euler characteristic -1

S.No.	SEM-Type	Exist (Yes/No)	Transitive or Not	Number of SEMs
1	$(3^5, 4)$	YES	No	3 (K_1, K_2, K_3)
2	$(3^4, 4^2)$	No	–	–
3	$(3^4, 8)$	No	–	–
4	$(3^2, 4, 3, 6)$	No	–	–
5	$(3, 4^4)$	No	–	–
6	$(3, 4, 8, 4)$	Yes	No	2 $(K_1(3, 4, 8, 4), K_2(3, 4, 8, 4))$
7	$(3, 6, 4, 6)$	No	–	–
8	$(4^3, 6)$	No	–	–
9	$(4, 6, 16)$	Yes	No	2 $(M_1(4, 6, 16), M_2(4, 6, 16))$
10	$(4, 8, 12)$	No	–	–
11	$(6^2, 8)$	Yes	No	2 $(N_1(6^2, 8), N_2(6^2, 8))$

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